J Eng Math (2007) 59:277–299
DOI 10.1007/s10665-006-9115-5

ORIGINAL PAPER

Elementary water waves

Francis Noblesse · **Chi Yang**

Received: 6 March 2006 / Accepted: 31 October 2006 / Published online: 10 January 2007

Abstract New elementary farfield and nearfield time-harmonic water waves, observed from a Galilean reference frame, are given. These two related classes of waves are modifications of classical elementary plane waves that are obtained in a simple way, using only elementary fundamental considerations, by analyzing waves that slowly grow from rest at time $T = -\infty$ and are bounded in the farfield. This approach circumvents the need for a radiation condition, which may indeed be regarded as (indirectly) satisfied ab initio by the elementary farfield and nearfield waves given here. The farfield waves are the product of classical elementary waves by a function, called radiation function, that is defined explicitly in terms of the dispersion function. Thus, this radiation function is valid not only for water waves, but more generally for a broad class of linear dispersive waves. For illustration and verification purposes, elementary stationary-phase considerations are used to determine the main properties (phase and group velocities, wave patterns, asymptote and cusp lines) of farfield waves, and two particular classes of water waves—steady ship waves and time-harmonic waves generated by an offshore structure—in uniform finite water depth are considered. The elementary nearfield waves can readily be used to construct free-surface Green functions or in a spectral representation of nearfield flows.

Keywords Dispersion relation · Dispersive waves · Radiation condition · Wave patterns · Water waves

1 Introduction

Waves are prevalent in physics and engineering, and accordingly have been extensively studied. They can be dispersive (e.g. water waves) or nondispersive (e.g. accoustic waves) and can be analyzed within a linear (small-amplitude waves) or nonlinear framework, using well-established alternative methods; e.g. [1,2].

F. Noblesse (✉)
NSWCCD, 9500 MacArthur Blvd, West Bethesda, MD 20817, USA
e-mail: francis.noblesse@navy.mil

C. Yang
College of Science, George Mason University, Fairfax, VA 22030, USA
e-mail: cyang@gmu.edu

C. Yang
Shanghai Jiao Tong University, Shanghai, China

Linear dispersive waves are considered here. A useful approach—widely used to analyze linear waves, notably dispersive waves of interest here—consists in considering time-harmonic waves, within the context of a Fourier analysis. This classical approach is based on Fourier superposition of elementary waves, typically obtained using separation of variables. Elementary waves—the basic building blocs in a Fourier analysis of dispersive waves—are the primary object of the present study. Although the approach expounded here is valid for a broad class of linear dispersive waves, water waves are considered for illustration purposes.

Thus, unsteady free-surface potential flows in water of uniform finite depth D are now considered. The Z-axis is vertical and points upward, and the mean free surface is taken as the plane $Z=0$. The sea floor $Z=-D$ is assumed to be a rigid wall. The flow is observed from a Galilean system of coordinates that advances at constant speed $\mathcal{U}$ along a direction chosen as the positive X-axis. The flow at a flow-field point $\mathbf{X}=(X,Y,Z)$ and time T is defined by a velocity potential $\Phi(\mathbf{X},T)$ that satisfies the Laplace equation

$$\Phi_{XX}+\Phi_{YY}+\Phi_{ZZ}=0 \tag{1a}$$

in the flow region, the sea-floor boundary condition

$$\Phi_Z=0 \quad \text{at } Z=-D\,, \tag{1b}$$

and—for small-amplitude waves—the linearized free-surface boundary condition

$$g\,\Phi_Z+(\partial_T-\mathcal{U}\,\partial_X)^2\,\Phi=0 \quad \text{at } Z=0\,. \tag{1c}$$

For typical free-surface flows about ships or other (finite) floating bodies, the potential $\Phi(\mathbf{X},T)$ vanishes in the horizontal farfield, i.e.,

$$|\nabla\Phi|\to 0 \quad \text{as } \sqrt{X^2+Y^2}\to\infty \tag{1d}$$

and satisfies a nearfield Neumann "body boundary condition"

$$\mathbf{n}\cdot\nabla\Phi=\text{given at body surface}\,. \tag{1e}$$

Here, $\mathbf{n}$ stands for a unit vector normal to the body surface. Finally, $\Phi(\mathbf{X},T)$ satisfies initial conditions

$$\Phi=\text{given} \quad \text{and} \quad \Phi_T=\text{given} \quad \text{at } T=0\,. \tag{1f}$$

Nondimensional coordinates $\mathbf{x}=(x,y,z)$, time t and flow potential ϕ are defined in terms of a reference length L, time $\sqrt{L/g}$ and potential $\sqrt{g\,L}\,L$. Here, g stands for the acceleration of gravity. Thus, one has

$$\mathbf{x}=\mathbf{X}/L \quad t=T\sqrt{g/L} \quad \phi=\Phi/(\sqrt{g\,L}\,L)$$

and the free-surface boundary condition (1c) becomes

$$\phi_z+(\partial_t-F\partial_x)^2\phi=0 \quad \text{at } z=0 \quad \text{with } F=\mathcal{U}/\sqrt{g\,L}\,. \tag{2}$$

The special case of established time-harmonic or steady flows, for which the initial conditions (1f) are irrelevant, is now considered. Flow about a ship that advances along a straight path, at constant speed $\mathcal{U}$, through regular waves or in calm water are conspicuous examples of the class of time-harmonic or steady free-surface flows under consideration. The potential of a time-harmonic flow is usually expressed as

$$\mathfrak{Re}\;\phi(\mathbf{x})\,\mathrm{e}^{\mathrm{i}ft} \quad \text{with } f=\omega\sqrt{L/g}\,. \tag{3}$$

Here, $\mathfrak{Re}$ stands for the real part, and ω is the frequency of the time-harmonic flow (as it appears in the moving system of coordinates). Steady flows correspond to the special case $f=0$ in (3) and hereafter. The spatial component $\phi(\mathbf{x})$ of the flow potential (3) satisfies the Laplace equation

$$\phi_{xx}+\phi_{yy}+\phi_{zz}=0 \tag{4a}$$

in the flow region, the sea-floor boundary condition

$$\phi_z=0 \quad \text{at } z=-d\,, \tag{4b}$$

the (linearized) free-surface boundary condition

$$\phi_z - f^2\phi - \mathrm{i}\,2\tau\,\phi_x + F^2\phi_{xx} = 0 \quad \text{at } z = 0\,, \tag{4c}$$

the farfield boundary condition

$$|\nabla\phi| \to 0 \quad \text{as } \sqrt{x^2+y^2} \to \infty\,, \tag{4d}$$

and a nearfield boundary condition

$$\mathbf{n}\cdot\nabla\phi = \text{given at body surface}\,. \tag{4e}$$

The free-surface condition (4c) readily follows from (2) and (3). The nondimensional water depth d in (4b), and the Froude number F, nondimensional frequency f and parameter τ in (4c) are defined as

$$d = D/L \quad F = \mathcal{U}/\sqrt{gL} \quad f = \omega\sqrt{L/g} \quad \tau = Ff = \mathcal{U}\,\omega/g\,. \tag{5}$$

The reference length L may be chosen at will. In particular, the choices $L = D$, $L = \mathcal{U}^2/g$, $L = g/\omega^2$ yield $d = 1$, $F = 1$, $f = 1$, respectively. One has $f = 0$ and $\tau = 0$ for steady flow about a ship advancing in calm water, and $F = 0$ and $\tau = 0$ for wave diffraction-radiation by an offshore structure.

The initial/boundary-value problem (1) and the boundary-value problem (4) are classical and have been widely considered in the literature. Basic mathematical aspects are considered in e.g. [3–7]. Green-function methods for solving the "ship-motion" boundary-value problem (4) for practical ship forms are reported in numerous studies; a partial list of references may be found in [8,9].

It is well known that a condition, known as a radiation condition, must be added to the boundary-value problem (4) to define a unique solution; e.g. [3–7]. This radiation condition, required to substitute for the loss of information associated with the fact that initial conditions are ignored in (4), can be enforced in a number of alternative ways, notably via a Green function that satisfies the free-surface boundary condition (4c). A good account of the substantial fundamental difficulties and practical complexities associated with the enforcement of a radiation condition to obtain a unique solution of the boundary-value problem (4), which corresponds to an established (time-harmonic or steady) flow of the form (3), is given in section 13γ of [6].

It is also well known that the initial/boundary-value problem (1) does not require a radiation condition to obtain a unique solution [5–7]. Thus, the need for a radiation condition is circumvented if the forcing term in the nearfield boundary condition (1e) is assumed to start from rest at some initial instant $t = 0$ and to reach a steady (time-harmonic or time-independent) state for t greater than some "building-up" time t_*, as explained in e.g. [6]. However, this approach involves the additional variable t and a transcient flow component (which depends on initial conditions that, in fact, are irrelevant, and vanishes as $t \to \infty$), and thus introduces unnecessary substantial complexities. An essentially similar, but considerably simpler, alternative way of circumventing the need for a radiation condition consists in considering the potential

$$\mathfrak{Re}\;\phi(\mathbf{x})\,e^{\varepsilon t + \mathrm{i} f t} = \mathfrak{Re}\;\phi(\mathbf{x})\,e^{\mathrm{i}(f - \mathrm{i}\varepsilon)t} \quad \text{with } 0 < \varepsilon \ll 1 \tag{6}$$

instead of the potential (3), which contains no information about initial conditions. The potential (6), employed for instance in [2,10,11], defines a flow that starts from rest at time $t = -\infty$. Thus, initial conditions are embedded in the potential (6).

The potential $\phi(\mathbf{x})$ in (6) satisfies the Laplace equation (4a), the sea-floor boundary condition (4b), the farfield and nearfield boundary conditions (4d) and (4e) and the modified free-surface boundary condition

$$\phi_z - f^2\phi - \mathrm{i}\,2\tau\,\phi_x + F^2\phi_{xx} + 2\,\mathrm{i}\,\varepsilon\,(f\phi + \mathrm{i}\,F\phi_x) = 0\,. \tag{7}$$

This free-surface condition, which readily follows from (2) and (6), differs from (4c) by a term that involves the "time-initialization" parameter ε in (6). A similar free-surface boundary condition can be formulated by invoking Rayleigh's artificial-viscosity technique; e.g. [6,12]. The artificial-viscosity technique and Lighthill's method, which assume a viscous fluid or a flow that slowly grows from rest at $t = -\infty$, are essentially equivalent. Thus, use of Rayleigh's or Lighthill's methods may arguably be considered a matter of personal preference. Lighthill's approach is used here because it is regarded as more direct and natural. Specifically, Lighthill's approach is preferred for three reasons: (a) it fundamentally addresses the fact that a radiation condition is required for the boundary-value problem (4) because the information associated with initial conditions is lost in this boundary-value problem, (b) it leads to a boundary-value problem that immediately follows from the initial/boundary-value problem (1), and (c) it does not require an artificial modification of the physical properties of the fluid and related nontrivial modifications of the boundary-value problem (e.g., the precise manner in which the free-surface boundary condition needs to be modified due to artificial viscosity effects is not a priori obvious). Expression (6) shows that Lighthill's approach is tantamount to considering a complex frequency $f - \mathrm{i}\,\varepsilon$ instead of a real frequency f in (3). Lighthill's approach, and the closely related and essentially equivalent "complex-frequency" and "artificial-viscosity" techniques, circumvent the need for a radiation condition, as already noted. Alternatively, these methods can be regarded as convenient indirect ways of satisfying a radiation condition.

As already noted, elementary waves associated with the boundary-value problem defined by the Laplace equation (4a), the sea-floor boundary condition (4b), the farfield and nearfield boundary conditions (4d) and (4e), and the free-surface boundary condition (7) are considered here. Thus, elementary waves that satisfy

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = 0 \quad \text{in } -d < z < 0\,, \tag{8a}$$

$$\phi_z = 0 \quad \text{at } z = -d\,, \tag{8b}$$

$$\phi_z - f^2\phi - \mathrm{i}\,2\tau\,\phi_x + F^2\phi_{xx} + 2\,\mathrm{i}\,\varepsilon\,(f\phi + \mathrm{i}\,F\phi_x) = 0 \quad \text{at } z = 0 \quad \text{with } 0 < \varepsilon \ll 1\,, \tag{8c}$$

$$|\nabla\phi| \to 0 \quad \text{as } \sqrt{x^2 + y^2} \to \infty \tag{8d}$$

are considered in this study (specifically, in Sects. 3 and 6). The nearfield boundary condition (4e) is ignored in the "farfield" boundary-value problem (8). As already noted, the farfield boundary-value problem (8) corresponds to the flow potential (6), which is null at the initial time $t = -\infty$. The initial conditions embedded in (6) and (8) and the farfield boundary condition (8d) can be expected to yield elementary wave solutions of (8a–8c) that are fully determined—except for the amplitudes of the waves, which can only be determined by considering the nearfield condition (4e), ignored here—with no need for further consideration, notably a radiation condition. In fact, the farfield boundary condition (8d) is shown further on to eliminate unacceptable elementary waves. This farfield condition indeed imposes a restriction that is essentially tantamount to enforcing a radiation condition, which is then effectively satisfied—in a simple and natural way—using Lighthill's approach associated with the potential (6).

2 Classical elementary waves

Classical elementary waves related to the boundary-value problem (4) are considered first. The classical elementary wave function

$$W(\mathbf{x}) = \mathrm{e}^{\,\mathrm{i}\,(\alpha x + \beta y)} \cosh\,[\,k\,(z + d)\,]/\cosh(k\,d) \tag{9a}$$

satisfies the sea-floor condition (4b), and satisfies the Laplace equation (4a) if

$$k = \sqrt{\alpha^2 + \beta^2}\,. \tag{9b}$$

The elementary wave (9a) also satisfies the free-surface boundary condition (4c) if the three parameters $\alpha\,,\beta$ and k satisfy the condition $\Delta = 0$, known as the dispersion relation, with the dispersion function Δ given by

$$\Delta = k\tanh(k\,d) - (f - F\alpha)^2. \tag{10}$$

In the deep-water limit $d \to \infty$, the elementary wave (9a) and the dispersion function (10) become

$$W(\mathbf{x}) = \mathrm{e}^{\,\mathrm{i}\,(\alpha x + \beta y) + k z} \qquad \Delta = k - (f - F\alpha)^2. \tag{11}$$

The deep-water dispersion function (11) is appreciably simpler than (10), which involves the additional water-depth parameter d and a hyperbolic function. The elementary waves defined by (3) and (9a) or (11) travel in a direction $(\alpha\,,\beta) = k\,(\cos\gamma, \sin\gamma)$ with speed, i.e., phase velocity

$$\mathbf{v}_p = \begin{Bmatrix} v_p^x \\ v_p^y \end{Bmatrix} = \frac{-f}{k^2}\begin{Bmatrix} \alpha \\ \beta \end{Bmatrix} = \frac{-f}{k}\begin{Bmatrix} \cos\gamma \\ \sin\gamma \end{Bmatrix}. \tag{12}$$

If (9b), which defines the wavenumber k in terms of the two Fourier variables α and β, is used in (10), the dispersion function $\Delta(\alpha,\beta,k;f,F,d)$ becomes a function $\Delta(\alpha,\beta;f,F,d)$ of the two Fourier variables α and β . The dispersion relation $\Delta(\alpha\,,\beta\,;f,F,d) = 0$ defines curve(s) in the Fourier plane $(\alpha\,,\beta)$. Specifically, the dispersion relation $\Delta = 0$ defines one curve (a circle) in the special case $F = 0$, and defines two or three curves if $F \neq 0$. The dispersion curves $\Delta(\alpha\,,\beta\,;f,F,d) = 0$ associated with the deep-water dispersion function (10) have been extensively considered in the literature; e.g. [11,13,14] .

The frequency f of dispersive waves, the wavenumbers α and β , the water depth d , and the speed F of the Galilean frame of reference from which the waves are observed are related by the dispersion relation. The group velocity for dispersive waves characterized by a dispersion relation $f = f(\alpha\,,\beta\,;F,d)$ is

$$\mathbf{v}_g = (v_g^x\,,v_g^y) = -(\partial f/\partial\alpha\,,\partial f/\partial\beta) = -(f_\alpha\,,f_\beta)\,;$$

see e.g. [1,2,6,7,12]. The water depth d and the speed F of the Galilean frame of reference from which the waves are observed are independent of the waves, and are not related to the wavenumbers α and β . Thus, if the dispersion relation is expressed as $\Delta(\alpha,\beta,f,F,d) = 0$, one has

$$\Delta_\alpha\,\delta\alpha + \Delta_\beta\,\delta\beta + \Delta_f\,\delta f + \Delta_F\,\delta F + \Delta_d\,\delta d = (\Delta_\alpha + \Delta_f\,f_\alpha)\,\delta\alpha + (\Delta_\beta + \Delta_f\,f_\beta)\,\delta\beta = 0$$

and the group velocity is given by

$$\mathbf{v}_g = \begin{Bmatrix} v_g^x \\ v_g^y \end{Bmatrix} = \frac{1}{\Delta_f}\begin{Bmatrix} \Delta_\alpha \\ \Delta_\beta \end{Bmatrix}. \tag{13}$$

Thus, the elementary wave (9), where α and β satisfy the dispersion relation $\Delta\,(\alpha\,,\beta\,;f,F,d) = 0$ defined by (10) with (9b), is an elementary solution that satisfies the Laplace equation and the boundary conditions (4b) and (4c) at the sea floor and the free surface. Accordingly, farfield waves can be represented by one-dimensional Fourier superpositions of elementary waves W along dispersion curves, i.e.,

$$\sum_{\Delta=0}\int_{\Delta=0} \mathrm{d}s\; a\,W. \tag{14}$$

In this Fourier representation of farfield waves, summation is performed over all the dispersion curves defined by the dispersion relation $\Delta\,(\alpha\,,\beta\,;f,F,d) = 0$, ds represents the differential element of arc length of a dispersion curve, W is the elementary wave (9) where the Fourier-plane point $(\alpha,\ \beta)$ lies on a dispersion curve, and a stands for a generic amplitude function (that can only be determined by satisfying a radiation condition and a nearfield boundary condition). No radiation condition, a critical (arguably the most important) farfield condition, is satisfied by the farfield waves defined by the classical Fourier representation (14). As a result, the representation (14) cannot yield correct wave patterns. For instance, this representation of farfield waves does not preclude steady ship waves ahead of a ship advancing in calm water. Thus, the representation (14), which satisfies the Laplace equation and the boundary conditions at the sea floor and the free surface but satisfies no radiation condition, is not a satisfactory farfield approximation.

3 Elementary farfield waves

Elementary waves associated with the complex frequency f–i ε in (6) are now considered. These elementary waves can be defined by considering the complex Fourier variables

$$\alpha + \mathrm{i}\,\varepsilon\,\alpha_1 \quad \beta + \mathrm{i}\,\varepsilon\beta_1 \quad k + \mathrm{i}\,\varepsilon\,k_1$$

in the elementary wave (9a), which thus becomes

$$e^{\mathrm{i}(\alpha x+\beta y)-\varepsilon(\alpha_1 x+\beta_1 y)}\cosh[(k+\mathrm{i}\,\varepsilon\,k_1)(z+d)\,]\,/\cosh[(k+\mathrm{i}\,\varepsilon\,k_1)\,d\,]. \tag{15a}$$

This elementary wave satisfies the sea-floor condition (8b), and satisfies the Laplace equation (8a) if the conditions

$$k^2=\alpha^2+\beta^2 \qquad k\,k_1=\alpha\,\alpha_1+\beta\,\beta_1 \qquad k_1^2=\alpha_1^2+\beta_1^2 \tag{15b}$$

are satisfied. These three conditions require

$$(\alpha_1,\beta_1,k_1)=P\,(\alpha,\beta,k)\,, \tag{15c}$$

where P stands for a proportionality factor between α_1,β_1,k_1 and α,β,k.

The elementary wave (15a) also satisfies the free-surface condition (8c) if the conditions

$$\Delta=0 \qquad k_1\,\Delta_k+\alpha_1\,\Delta_\alpha-\Delta_f=0 \tag{16}$$

are satisfied. Here, $\Delta_k=\tanh(k\,d)+k\,d/\cosh^2(k\,d)$ and $\Delta_\alpha=2\,(\tau-F^2\alpha)$ stand for the derivatives of the dispersion function $\Delta(\alpha\,,\beta\,,k\,;f,F,d)$ defined by (10) with respect to k and α , and

$$\Delta_f=-\,2\,(f-F\alpha) \tag{17}$$

is the derivative of the dispersion function Δ with respect to the frequency f. If the second of the three relations (15b) is used in (16), one obtains

$$\Delta=0 \qquad \alpha_1\,\Delta_\alpha+\beta_1\,\Delta_\beta-\Delta_f=0\,, \tag{18}$$

where Δ_α and Δ_β now stand for the derivatives

$$\Delta_\alpha=[\tanh(k\,d)+k\,d/\cosh^2(k\,d)]\,\alpha/k+2\,(\tau-F^2\alpha)\,, \tag{19a}$$

$$\Delta_\beta=[\tanh(k\,d)+k\,d/\cosh^2(k\,d)]\,\beta/k \tag{19b}$$

of the dispersion function $\Delta(\alpha\,,\beta\,;f,F,d)$ defined by (10) and (9b) with respect to the Fourier variables α and β . The dispersion relations (18) correspond to the limit $\varepsilon\to 0$ of the dispersion relation

$$\Delta(\alpha+\mathrm{i}\,\varepsilon\,\alpha_1,\ \beta+\mathrm{i}\,\varepsilon\,\beta_1\,;f-\mathrm{i}\,\varepsilon\,,F,d)=0 \tag{20}$$

associated with the potential (6) and the elementary wave (15a), as expected.

Expressions (15c) and (18) define P as

$$P=\Delta_f/(\alpha\,\Delta_\alpha+\beta\,\Delta_\beta)=\Delta_f/(k\Delta_k)\,,$$

where $\Delta_k=\Delta_\alpha\,\alpha/k+\Delta_\beta\,\beta/k$ now—and hereafter—stands for the derivative of the dispersion function $\Delta(\alpha,\,\beta\,;f,F,d)$ in the radial direction $(\alpha,\beta)/k$. Expressions (19) yield

$$\Delta_k=\tanh(k\,d)+k\,d/\cosh^2(k\,d)+2\,(\tau-F^2\alpha)\,\alpha/k\,. \tag{21}$$

The foregoing expression for P yields $P=\mu/Q$ with $Q=k\,|\Delta_k|/|\Delta_f|\geq 0$ and

$$\mu=\mathrm{sign}\Delta_f\,\mathrm{sign}\Delta_k\,. \tag{22a}$$

Thus, the elementary wave function defined by (15a) with (15c) becomes

$$e^{\mathrm{i}\,(\alpha x+\beta y)-\mu\,\varepsilon\,(\alpha x+\beta y)/Q}\cosh[\,k\,(1+\mathrm{i}\,\mu\,\varepsilon/Q)(z+d\,)\,]/\cosh[\,k\,(1+\mathrm{i}\,\mu\,\varepsilon/Q)\,d\,]. \tag{22b}$$

The elementary wave (22), where $0<\varepsilon\ll 1$ and the Fourier-plane point (α,β) lies on a dispersion curve $\Delta=0$, satisfies the Laplace equation and the boundary conditions at the sea floor and the free surface.

The elementary wave (22b) also satisfies the farfield condition (8d) if $0<\mu\,\mathrm{sign}(\alpha\,x+\beta\,y)$, since one has $0<\varepsilon$ and $0\leq Q$. Expression (22a) for μ and the polar representations

$$(\,x\,,y\,)=h\,(\,\cos\theta\,,\sin\theta\,) \quad (\,\alpha\,,\beta\,)=k\,(\,\cos\gamma\,,\sin\gamma\,) \tag{23}$$

of the horizontal coordinates x, y and the Fourier variables α, β then show that bounded elementary waves are obtained if the condition

$$\mathrm{sign}\Delta_f \,\mathrm{sign}\Delta_k = \mathrm{sign}(\alpha\, x+\beta\, y) = \mathrm{sign}\cos(\gamma-\theta)$$

is satisfied. This condition yields

$$\begin{Bmatrix} \theta-\pi/2 < \gamma < \theta+\pi/2 \\ \theta+\pi/2 < \gamma < \theta+3\pi/2 \end{Bmatrix} \quad \text{if} \quad \begin{Bmatrix} 0 < \mathrm{sign}\Delta_f \,\mathrm{sign}\Delta_k \\ \mathrm{sign}\Delta_f \,\mathrm{sign}\Delta_k < 0 \end{Bmatrix}. \tag{24}$$

The condition (24) defines "acceptable" sections, which depend on the angle θ, of a dispersion curve $\Delta = 0$ in the classical representation (14) of farfield waves. Alternatively, (14) can be modified as

$$\sum_{\Delta=0} \int_{\Delta=0} \mathrm{d}s\, A\, \mathcal{R}^F W, \tag{25a}$$

where W is the elementary wave (9), A stands for a generic amplitude function, and $\mathcal{R}^F$ is defined as

$$\mathcal{R}^F = 1 + \mathrm{sign}\Delta_f \,\mathrm{sign}\Delta_k \,\mathrm{sign}(\alpha\, x+\beta\, y)\,. \tag{25b}$$

The function $\mathcal{R}^F$ is called farfield radiation function hereafter. The representation (14), where the dispersion curves $\Delta = 0$ are restricted in accordance with (24), and the alternative representation (25) restrict the dispersion curves in equivalent ways. The amplitude functions A in (25a) and a in (14) are (obviously) not identical (in fact, one has $a = A\,\mathcal{R}^F$). The generic amplitude function A in (25a) can only be determined if the nearfield boundary condition (4e) is considered; however, this nearfield boundary condition is ignored in the farfield boundary-value problem (8) considered here, as already explained in the introduction.

For instance, in the special case of wave diffraction-radiation by an offshore structure, one has $F = 0$ and $\tau = 0$, and (10), (17) and (21) become

$$\Delta = k\tanh(k\,d) - f^2 \quad \Delta_f = -2f < 0 \quad 0 < \Delta_k = \tanh(k\,d) + k\,d/\cosh^2(k\,d)\,. \tag{26}$$

The dispersion relation $\Delta = 0$ then yields

$$1/k^\omega = \tanh(d^\omega k^\omega) \quad \text{with } d^\omega = f^2 d = D\,\omega^2/g \quad \text{and } k^\omega = k/f^2 = Kg/\omega^2. \tag{27a}$$

Here, Eq. 5 was used. The dispersion relation (27a) defines a single dispersion curve, namely the circle

$$k^\omega = k_0^\omega(d^\omega), \tag{27b}$$

where k_0^ω is the root of the dispersion relation (27a). Expressions (26) and (23) then show that the radiation function $\mathcal{R}^F$ given by (25b) is null along the "dead" half $\theta-\pi/2 < \gamma < \theta+\pi/2$ of the dispersion circle (27b), in agreement with (24). The "alive" and "dead" halves of the dispersion circle (27b) that correspond to $\theta = \pi/4$ are shown in Fig. 1 . A deep-water approximation and a shallow-water approximation, given in Appendix 1, to the root $1/k_0^\omega$ of the dispersion relation (27a) are

$$1/k_0^\omega \sim 1 \quad \text{as } d^\omega \to \infty \qquad 1/k_0^\omega \sim \sqrt{d^\omega}\Big/\left[1+\left(\frac{1}{6}+\frac{11}{360}\,d^\omega\right)d^\omega\right] \quad \text{as } d^\omega \to 0. \tag{28}$$

These asymptotic approximations and the root of (27a) are depicted in Fig. 2. This figure shows that time-harmonic water waves (without forward speed) are shorter in finite water depth than in deep water, as is well known.

The radiation function $\mathcal{R}^F$ in (25a), associated with the potential (6) and the related dispersion relation (20), introduces restrictions that do not exist in the classical representation (14). The restrictions introduced by the radiation function $\mathcal{R}^F$ account for the information that is included in the potential (6) and the related dispersion relation (20), but is lost if the time-initialization parameter ε is set null in (6) and (20). Specifically, the potential (6) assumes a flow that starts from rest at time $t = -\infty$, whereas the potential (3) includes no information about initial conditions, as already noted. Accordingly, the potential (3) and the

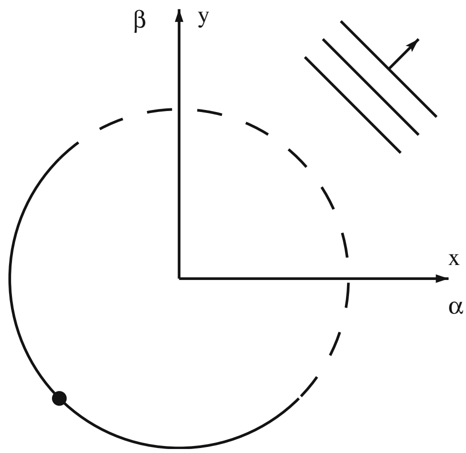

Fig. 1 Dead (dashed line) and alive (solid line) portions of dispersion circle, in Fourier plane $(\alpha\,,\beta)$, that correspond to time-harmonic water waves generated in direction $\theta = \pi/4$ in physical space $(x\,,y)$. Large dot on dispersion circle, at angle $\gamma = 5\pi/4$, identifies main generator of farfield waves

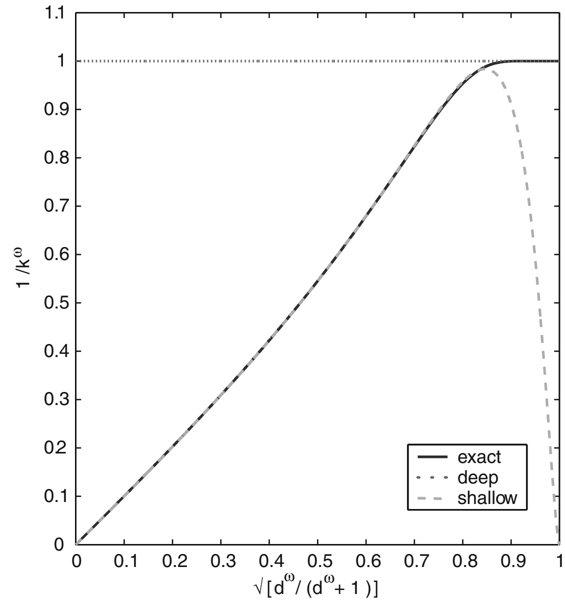

Fig. 2 Variation of root $1/k^{\omega}$ of dispersion relation (27a), and of related deep-water and shallow-water approximations (28), for wave diffraction-radiation by an offshore structure in finite water depth, in terms of water depth $0 \le \sqrt{d^{\omega}/(d^{\omega}+1)} \le 1$

related dispersion relation cannot yield uniquely defined time-harmonic or steady flows; indeed, if (3) is used, uniquely defined flows can only be defined by invoking a radiation condition that specifies that wave energy is radiated away from the wave generator [3–7]. However, if the potential (6) is used, the need for a radiation condition is avoided (Alternatively, as already noted in the introduction, a radiation condition may be regarded as effectively satisfied a priori, with no need for further consideration, by disallowing unbounded farfield waves in the foregoing analysis of elementary waves that slowly grow from rest at time $t = -\infty$). The Fourier representation of farfield waves (25) has been obtained here in a simple and natural manner, using only elementary considerations and analysis based on the potential (6) and the related farfield boundary-value problem (8). Expression (25b) for the radiation function $\mathcal{R}^F$ in (25a) is explicitly defined in terms of the derivative Δ_f of the dispersion function Δ with respect to the frequency f and the derivative of Δ in the radial direction $(\alpha,\beta)/k$. Thus, the Fourier superposition (25) of farfield waves is readily applicable to a broad class of linear dispersive waves characterized by a dispersion function Δ . Indeed, multiplication of an elementary wave function W (that can typically be obtained using separation of variables) associated with a given dispersive medium by the radiation function (25b), as in (25a), circumvents the need for enforcing a radiation condition (which may be regarded as effectively satisfied, ab initio).

4 Farfield wave patterns

As already noted, no radiation condition is satisfied by the Fourier superposition (14) of classical elementary waves, which therefore cannot yield correct farfield wave patterns; for instance, Eq. 14 does not preclude steady ship waves ahead of a ship advancing in calm water. However, the Fourier superposition (25) of modified farfield waves $\mathcal{R}^F W$, for which a "built-in" radiation condition is (indirectly) satisfied, can be expected to yield correct farfield waves and wave patterns, as verified below. Specifically, farfield waves are easily determined using asymptotic analysis, based on Kelvin's method of stationary phase, of the Fourier integral (25a). In fact, elementary stationary-phase considerations suffice to determine the main properties of farfield waves (phase and group velocities, wave patterns, asymptote and cusp lines), and to illustrate the fundamental relationships that exist between farfield waves in the physical plane $(x\,,y)$ and the dispersion function $\Delta(\alpha\,,\beta\,;f,F,d)$ and related dispersion curves $\Delta=0$ in the Fourier plane $(\alpha\,,\beta)$.

In the farfield $h\to\infty$, the main contribution to the Fourier integral (25a) stems from points where the phase $\varphi=\alpha\,x+\beta\,y$ of the trigonometric function $\mathrm{e}^{\,\mathrm{i}\,\varphi}$ in (9a) is stationary. A point $(\alpha\,,\beta)$ of a dispersion curve $\Delta(\alpha\,,\beta\,;f,F,d)=0$ where the phase φ is stationary is determined by the relation

$$\mathrm{d}\varphi/\mathrm{d}s=x\,\mathrm{d}\alpha/\mathrm{d}s+y\,\mathrm{d}\beta/\mathrm{d}s=0\,.$$

Here, $\mathrm{d}s$ is the differential element of arc length of a dispersion curve. The vectors $(\mathrm{d}\alpha/\mathrm{d}s,\mathrm{d}\beta/\mathrm{d}s)$ and $\nabla\,\Delta=(\Delta_\alpha,\Delta_\beta)$ are tangent and normal to a dispersion curve, respectively, and therefore are orthogonal. Thus, the vector $(x\,,y)$ is normal to a dispersion curve at a point (α,β) where $\mathrm{d}\varphi/\mathrm{d}s=0$, and the relation

$$(x\,,y)/\sqrt{x^2+y^2}=\nu\,(\Delta_\alpha\,,\Delta_\beta)/\sqrt{\Delta_\alpha^2+\Delta_\beta^2}\quad\text{with }\nu=\pm1$$

holds at such a point. This stationary-phase relation yields

$$\mathrm{sign}(\alpha\,x+\beta\,y)=\nu\,\mathrm{sign}(\alpha\,\Delta_\alpha+\beta\,\Delta_\beta)=\nu\,\mathrm{sign}\Delta_k.$$

Expression (25b) then yields $\mathcal{R}^F=1+\nu\;\mathrm{sign}\Delta_f$ at a point of stationary phase. Thus, a point of stationary phase yields a nonzero contribution to the Fourier integral (25a) only if $\nu=\mathrm{sign}\Delta_f$, i.e., if

$$\mathrm{sign}(\alpha\,x+\beta\,y)=\mathrm{sign}\Delta_f\,\mathrm{sign}\Delta_k=\mu \tag{29}$$

and

$$\frac{(x\,,y)}{\sqrt{x^2+y^2}}=(\cos\theta\,,\sin\theta)=\mathrm{sign}\Delta_f\,\frac{(\Delta_\alpha\,,\Delta_\beta)}{\sqrt{\Delta_\alpha^2+\Delta_\beta^2}}=\mathrm{sign}\Delta_f\,\frac{\nabla\,\Delta}{\|\nabla\,\Delta\|}\,. \tag{30a}$$

The condition (30a) shows that a point $(\alpha\,,\beta)$ of a dispersion curve (in the Fourier plane) mostly generates waves (in the physical space) in a direction

$$\tan\theta=y/x=\Delta_\beta/\Delta_\alpha=\tan\delta \tag{30b}$$

that is orthogonal to the dispersion curve and oriented as $(\mathrm{sign}\Delta_f)\,\nabla\,\Delta$. Here, θ is the angle between the ray (x,y) and the x-axis in the physical space, and δ is the angle between the vector $\nabla\,\Delta=(\Delta_\alpha,\Delta_\beta)$ normal to a dispersion curve $\Delta(\alpha,\beta;f,F,d)=0$ and the α-axis. Conversely, farfield waves observed in a direction θ stem mostly from the point(s) of the dispersion curve(s) where the condition (30a) holds. Expressions (30a) and (13) yield

$$\frac{(x\,,y)}{\sqrt{x^2+y^2}}=\frac{|\Delta_f|}{\|\nabla\Delta\|}\,\left(v_{\mathrm{g}}^x\,,v_{\mathrm{g}}^y\right). \tag{30c}$$

Thus, as expected, waves are generated in the direction of the group velocity $\mathbf{v}_{\mathrm{g}}$, i.e., the direction along which energy is transmitted. The results expressed by (30) agree with [2,14,15].

For instance, for wave diffraction-radiation by an offshore structure, (19) and (21), with $F=0$ and $\tau=0$, and (23) yield

$$\Delta_\alpha/\Delta_k=\alpha/k=\cos\gamma\qquad\Delta_\beta/\Delta_k=\beta/k=\sin\gamma. \tag{31}$$

These relations, (30a) and (26), then yield

$$\theta=\gamma+\pi\,. \tag{32}$$

Thus, a point γ of the dispersion circle (27) for wave diffraction-radiation by an offshore structure mostly generates waves in a direction $\theta=\gamma+\pi$. Conversely, farfield waves observed in a direction θ are mostly generated by the point $\gamma=\theta-\pi$ of the dispersion circle. For instance, farfield waves in a direction $\theta=\pi/4$ are generated at $\gamma=-3\pi/4$, as shown in Fig. 1 . Expressions (12) and (13) with (32), (31) and (26) define the phase and group velocities as

$$\mathbf{v}_{\mathrm{p}}=\begin{Bmatrix} v_{\mathrm{p}}^{x} \\ v_{\mathrm{p}}^{y} \end{Bmatrix}=v_{\mathrm{p}}\begin{Bmatrix} \cos\theta \\ \sin\theta \end{Bmatrix} \quad \text{with} \quad f\,v_{\mathrm{p}}=v_{\mathrm{p}}^{\omega}=\frac{1}{k_0^{\omega}}\,, \tag{33a}$$

$$\mathbf{v}_{\mathrm{g}}=\begin{Bmatrix} v_{\mathrm{g}}^{x} \\ v_{\mathrm{g}}^{y} \end{Bmatrix}=v_{\mathrm{g}}\begin{Bmatrix} \cos\theta \\ \sin\theta \end{Bmatrix} \quad \text{with} \quad f\,v_{\mathrm{g}}=v_{\mathrm{g}}^{\omega}=\frac{2\,d^{\omega}k_0^{\omega}+\sinh(2\,d^{\omega}k_0^{\omega})}{4\,\cosh^2(d^{\omega}k_0^{\omega})}\,. \tag{33b}$$

Here, $v^{\omega}=f\,v=V\omega/g$ and k_0^{ω} is the root of the dispersion relation (27a). Thus, one obtains the well-known result

$$\frac{1}{2}\le\frac{v_{\mathrm{g}}^{\omega}}{v_{\mathrm{p}}^{\omega}}=\frac{1}{2}+\frac{d^{\omega}k_0^{\omega}}{\sinh(2\,d^{\omega}k_0^{\omega})}\le 1\,. \tag{33c}$$

Expressions (33a) and (33b) and the approximations (28) yield $v_{\mathrm{p}}^{\omega}\to 1$ and $v_{\mathrm{g}}^{\omega}\to 1/2$ as $d^{\omega}\to\infty$, and $v_{\mathrm{p}}^{\omega}\sim\sqrt{d^{\omega}}$ and $v_{\mathrm{g}}^{\omega}\sim\sqrt{d^{\omega}}$ as $d^{\omega}\to 0$, as is well known.

Expressions (25b) and (29) show that one has $\mathcal{R}^F=2$ at a point of stationary phase, where (30a) yields $\text{sign}(x\,\Delta_{\alpha}+y\,\Delta_{\beta})=\text{sign}\Delta_f$. Thus, the radiation function (25b) and the alternative function

$$\mathcal{R}^F=1+\text{sign}\Delta_f\,\text{sign}(x\,\Delta_{\alpha}+y\,\Delta_{\beta}) \tag{34}$$

yield identical farfield waves. Indeed, the flows defined by (25a), with the farfield radiation function $\mathcal{R}^F$ given by (25b) or (34), differ by

$$\sum_{\Delta=0}\int_{\Delta=0}\mathrm{d}s\,A\,(\text{sign}\Delta_f)\,[\text{sign}\Delta_k\,\text{sign}(\alpha\,x+\beta\,y)-\text{sign}(x\,\Delta_{\alpha}+y\Delta_{\beta})]\,W. \tag{35}$$

Expressions (29) and (30a) yield $\text{sign}\Delta_k\,\text{sign}(\alpha\,x+\beta\,y)=\text{sign}\Delta_f=\text{sign}(x\,\Delta_{\alpha}+y\,\Delta_{\beta})$ at a point of stationary phase, where the integrand of (35) is then null and continuous. It follows that points of stationary phase in the Fourier superposition (25a) of elementary waves, with the radiation function $\mathcal{R}^F$ given by (25b) or (34), yield identical farfield waves as previously noted. Expression (34) was obtained in [16] via a more complicated approach based on a farfield asymptotic analysis of a generic Green function, given by a singular double Fourier integral, that accounts for nearfield effects. Thus, an equivalent representation of farfield waves has been obtained here in a simpler and more direct manner, using only elementary analysis, from the Laplace equation, the boundary conditions at the sea floor and the free surface, and the farfield boundary condition.

Additional information about important features of farfield waves can be obtained from the dispersion function $\Delta(\alpha,\beta;f,F,d)$ and the related dispersion curves $\Delta=0$. Farfield wave patterns are now considered. The relations (30a) yield

$$\begin{Bmatrix} x \\ y \end{Bmatrix}=\text{sign}\Delta_f\,\frac{h}{\|\nabla\,\Delta\|}\begin{Bmatrix} \Delta_{\alpha} \\ \Delta_{\beta} \end{Bmatrix} \quad \text{with}\ h=\sqrt{x^2+y^2}\,.$$

Thus, the phase $\varphi=\alpha\,x+\beta\,y$ of the trigonometric function $\mathrm{e}^{\,\mathrm{i}\varphi}$ in (9a) and (25a) is given by

$$\varphi=\text{sign}\Delta_f\,\frac{h}{\|\nabla\,\Delta\|}(\alpha\,\Delta_{\alpha}+\beta\,\Delta_{\beta})=\text{sign}\Delta_f\,\frac{h\,k\,\Delta_k}{\|\nabla\,\Delta\|}\,.$$

This relation yields $h/\|\nabla\,\Delta\|=|\varphi|/|k\,\Delta_k|$, and x and y can be expressed as

$$\begin{Bmatrix} x \\ y \end{Bmatrix}=|\varphi|\,\frac{\text{sign}\Delta_f}{k\,|\Delta_k|}\begin{Bmatrix} \Delta_{\alpha} \\ \Delta_{\beta} \end{Bmatrix}. \tag{36}$$

These relations, where the point $(\alpha\,,\beta)$ is allowed to move along a given dispersion curve $\Delta = 0$, yield parametric equations for the coordinates x and y associated with a specified value of the phase φ, e.g. a particular wave crest or trough. If the dispersion relation $\Delta = 0$ defines several dispersion curves, as is the case for the dispersion function (10) if $F \neq 0$, different waves are obtained as the point $(\alpha\,,\beta)$ in (36) is allowed to move along every dispersion curve.

Successive waves, e.g. a series of wave crests, are obtained if a series of phase values $\varphi_n = 2\,n\,\pi$ with $n = 1, 2, 3, \ldots$, are considered in (36).Thus, farfield wave patterns are readily constructed from the dispersion function Δ using the parametric equations

$$\frac{1}{2\,n\,\pi}\begin{Bmatrix} x_n \\ y_n \end{Bmatrix} = \frac{\text{sign}\Delta_f}{k\,|\Delta_k|}\begin{Bmatrix} \Delta_\alpha \\ \Delta_\beta \end{Bmatrix} = \frac{\text{sign}\Delta_f\ \text{sign}\Delta_k}{\alpha\,\Delta_\alpha + \beta\,\Delta_\beta}\begin{Bmatrix} \Delta_\alpha \\ \Delta_\beta \end{Bmatrix}. \tag{37}$$

Expressions (37) and (13) yield

$$\mathbf{v}_{\rm g} = \begin{Bmatrix} v_{\rm g}^x \\ v_{\rm g}^y \end{Bmatrix} = \frac{k\,|\Delta_k|}{2\,n\,\pi\,|\Delta_f|}\begin{Bmatrix} x_n \\ y_n \end{Bmatrix} = \frac{k\,|\Delta_k|}{2\pi\,|\Delta_f|}\,\frac{\mathbf{x}_n}{n}\,. \tag{38}$$

This relation shows that wave energy is transmitted (at the group velocity $\mathbf{v}_{\rm g}$) along radial lines that radiate from the wave generator (i.e., the origin of the wave pattern), as expected and in agreement with (30c). As expected, the phase velocity $\mathbf{v}_{\rm p}$ is orthogonal to the constant-phase curves (e.g. wave crests and troughs) defined by (37); see Appendix 2. An illustration of these well-known properties of the group and phase velocities is given in [14] for time-harmonic ship waves in deep water.

For instance, for wave diffraction-radiation by an offshore structure, (37), (31), (26) and (27a) yield

$$\begin{Bmatrix} x_n^\omega \\ y_n^\omega \end{Bmatrix} = f^2\begin{Bmatrix} x_n \\ y_n \end{Bmatrix} = \frac{-2\,n\,\pi}{k_0^\omega}\begin{Bmatrix} \cos\gamma \\ \sin\gamma \end{Bmatrix} \quad \text{with } -\pi \le \gamma \le \pi, \tag{39}$$

where k_0^ω is the root of the dispersion relation (27a). The parametric equations (39) define a series of circular waves $h_n^\omega = 2\,n\,\pi/k_0^\omega$. These waves propagate outward, in accordance with (33a). Expressions (33) show that both the phase velocity $\mathbf{v}_{\rm p}$ and the group velocity $\mathbf{v}_g$ are orthogonal to the waves, as can be expected from the foregoing results in the particular case of circular waves.

The parametric equations (37) readily yield

$$x_n^2 + y_n^2 \to \infty \quad \text{if } \Delta_k = 0\,. \tag{40}$$

Thus, a point $(k\,,\gamma)$ of a dispersion curve where Δ_k vanishes yields an asymptote of the farfield wave pattern, at an angle θ that is orthogonal to the angle γ in the polar representation (23) at which $\Delta_k = 0$. "Asymptote points" $\Delta_k = 0$ are considered further in Appendix 3. Now consider a dispersion curve with an inflection point (α^i, β^i), i.e., (see Appendix 4) a point where

$$\Delta_\beta^2\,\Delta_{\alpha\alpha} - 2\,\Delta_\alpha\,\Delta_\beta\,\Delta_{\alpha\beta} + \Delta_\alpha^2\,\Delta_{\beta\beta} = 0\,. \tag{41}$$

As the point $(\alpha\,,\beta)$ moves along the dispersion curve in the vicinity of the inflection point (α^i, β^i), the angle δ between the vector $(\Delta_\alpha\,,\Delta_\beta)$ normal to the dispersion curve and the α-axis reaches a local maximum or minimum δ^i at the inflection point. Accordingly, the corresponding ray angle θ in the physical space also reaches a local maximum or minimum θ^i, given by $\theta^i = \delta^i$ or $\theta^i = \delta^i + \pi$, and the farfield wave pattern has a cusp at the ray θ^i.

Thus, asymptote and cusp lines of a wave pattern correspond to roots of the equation $\Delta_k = 0$ and to inflection points of the dispersion curves $\Delta = 0$, respectively. Cusp lines have been widely considered in the literature. However, asymptote lines are generally overlooked, or not distinguished from cusp lines. These simple results and expressions (30), (37) and (38) show that considerable information about farfield dispersive waves can be obtained—using only elementary considerations—from the dispersion function $\Delta(\alpha\,,\beta\,;f, F, d)$. Detailed illustrations of the relationship between the dispersion function and the related dispersion curves $\Delta = 0$ in the Fourier plane $(\alpha\,,\beta)$ and the corresponding farfield waves in the physical plane $(x\,,y)$ are given in [14,15] for diffraction-radiation by a ship advancing through regular waves in deep water.

5 Illustrative application to steady ship waves in finite water depth

Diffraction-radiation of regular waves by an offshore structure, which corresponds to the special case $F = 0$ in (10), has already been considered above. The special case $f = 0$, which corresponds to steady ship waves, is now considered for further illustration and verification. The dispersion function (10) becomes

$$\Delta = k\tanh(k\,d) - F^2\alpha^2 = k\tanh(k\,d) - F^2k^2\cos^2\gamma\,. \tag{42}$$

This dispersion function is an even function of both α and β. Thus, the dispersion curves $\Delta = 0$ are symmetric with respect to the axis $\alpha = 0$ and the axis $\beta = 0$.

The dispersion function (42) shows that dispersion curves $\Delta = 0$ are defined by

$$k^{U\gamma} = \tanh(d^{\,U\gamma}k^{U\gamma}) \quad \text{with} \begin{Bmatrix} d^{\,U\gamma} = d^{\,U}/\cos^2\gamma \\ k^{U\gamma} = k^U\cos^2\gamma \end{Bmatrix} \text{ and } \begin{Bmatrix} d^{\,U} = d/F^2 = D\,g/\mathcal{U}^2 \\ k^U = F^2k = K\mathcal{U}^2/g \end{Bmatrix}. \tag{43}$$

The dispersion relation (43) defines the normalized wavenumber $k^{U\gamma}$ in terms of the normalized water depth $d^{\,U\gamma}$. This equation has no positive real root if $d^{\,U\gamma} < 1$ or one positive real root if $1 < d^{\,U\gamma}$. Waves therefore exist for all values of γ if $1 < d^{\,U}$. However, waves exist only for $|\cos\gamma| < \sqrt{d^{\,U}}$ if $d^{\,U} < 1$. A deep-water approximation and a shallow-water approximation, given in Appendix 5, to the root $k^{U\gamma}$ of the dispersion relation (43) are

$$k^{U\gamma} \sim 1 \quad \text{as } d^{\,U\gamma} \to \infty \qquad k^{U\gamma} \sim \frac{\sqrt{3\eta}}{d^{\,U\gamma}}\left[1 + \left(\frac{3}{5} + \frac{93}{175}\eta\right)\eta\right] \quad \text{as } d^{\,U\gamma} \to 1 \quad \text{with } \eta = 1 - \frac{1}{d^{\,U\gamma}}\,. \tag{44}$$

These asymptotic approximations and the root of (43) are depicted in Fig. 3. This figure shows that steady ship waves are longer in finite water depth than in deep water, as is well known.

The dispersion function (42) and the relations $\beta^2 = k^2 - \alpha^2$ show that, at a dispersion curve $\Delta = 0$, the speed-scaled Fourier variables $\alpha^U = F^2\alpha$ and $\beta^{\,U} = F^2\beta$ are defined in terms of the speed-scaled wavenumber $k^U = F^2k$ by the parametric equations

$$\alpha^U = \pm\sqrt{k^U\tanh(d^{\,U}k^U)} \quad \beta^{\,U} = \pm\,k^U\sqrt{1 - \tanh(d^{\,U}k^U)/k^U} \quad \text{with } k_0^U \le k^U, \tag{45a}$$

where the smallest wavenumber $k_0^U(d^{\,U})$ is given by

$$k_0^U = \begin{Bmatrix} \text{root of } k_0^U = \tanh(d^{\,U}k_0^U) \\ 0 \end{Bmatrix} \text{ if } \begin{Bmatrix} 1 < d^{\,U} \\ d^{\,U} \le 1 \end{Bmatrix}. \tag{45b}$$

The parametric equations (45) yield

$$\alpha^U \sim \pm\sqrt{k^U} \quad \beta^{\,U} \sim \pm\,k^U\sqrt{1 - 1/k^U} \quad \text{with } 1 \le k^U \quad \text{as } d^{\,U} \to \infty\,, \tag{46a}$$

$$\begin{Bmatrix} \alpha^U \sim \pm\sqrt{d^{\,U}}\,k^U\sqrt{1 - (d^{\,U})^2(k^U)^2/3} \\ \beta^{\,U} \sim \pm\,k^U\sqrt{1 - d^{\,U} + (d^{\,U})^3(k^U)^2/3} \end{Bmatrix} \quad \text{with } 0 \le d^{\,U} \le 1 \quad \text{as } k^U \to 0\,. \tag{46b}$$

The asymptotic approximations (46b) yield

$$\beta^{\,U}/\alpha^{\,U} \sim \pm\sqrt{1/d^{\,U} - 1} \quad \text{with } 0 \le d^{\,U} \le 1 \quad \text{as } k^U \to 0\,.$$

Thus, at the origin $k^U = 0$, the dispersion curve is tangent to the axis $\beta^{\,U} = 0$ if $d^{\,U} = 1$, or the axis $\alpha^U = 0$ if $d^{\,U} = 0$. In fact, the parametric equations (45) show that the dispersion curve in the "zero-depth" case $d^{\,U} = 0$ is the axis $\alpha^U = 0$. The dispersion curves defined by the parametric equations (45) are depicted in

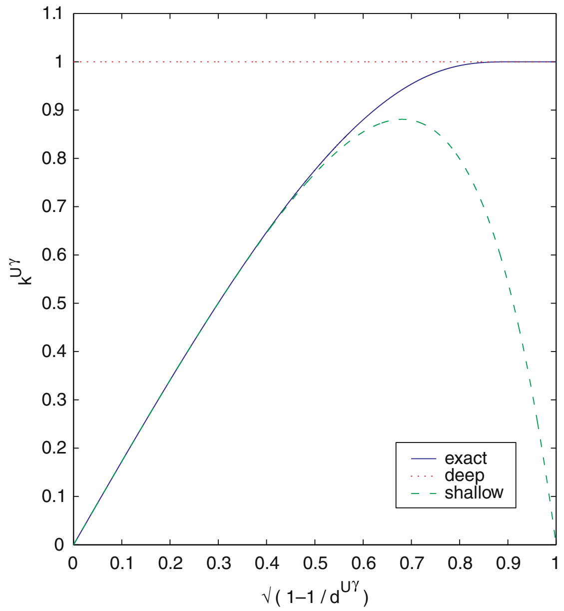

Fig. 3 Steady ship waves in finite water depth: variation of root $k^{U\gamma}$ of dispersion relation (43), and of related deep-water and shallow-water approximations (44), in terms of water depth $0 \le \sqrt{1-1/d^{U\gamma}} \le 1$ with $1 \le d^{U\gamma}$

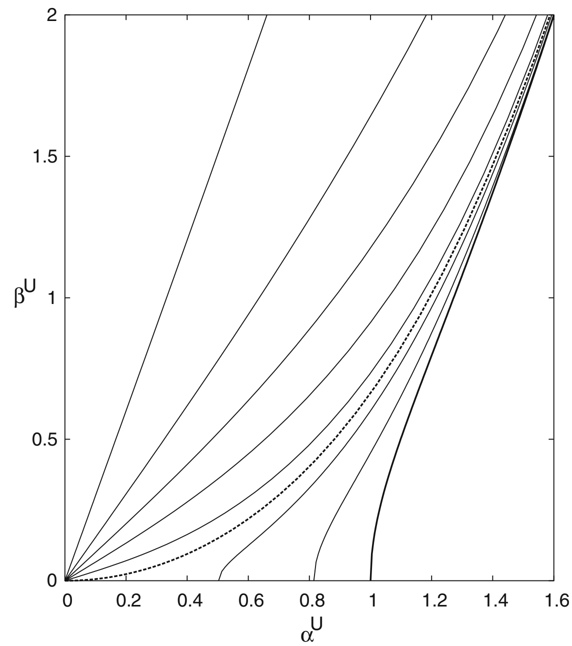

Fig. 4 Steady ship waves in finite water depth: dispersion curves for $d^U = \infty$ (thick solid line), 1.4 , 1.1 , 1 (thick dashed line), 0.9 , 0.7 , 0.5 , 0.3 and 0.1 in region $0 \le \alpha^U \le 1.6$ and $0 \le \beta^U \le 2$ of the speed-scaled Fourier variables α^U and β^U

Fig. 4 in the region $0 \le \alpha^U \le 1.6$, $0 \le \beta^U \le 2$ of the speed-scaled Fourier variables α^U and β^U for $d^U = \infty$, 1.4 , 1.1 , 1 , 0.9 , 0.7 , 0.5 , 0.3 and 0.1 . In view of the symmetry of the dispersion curves with respect to α and β, only the quadrant $0 \le \alpha^U, 0 \le \beta^U$ is considered in Fig. 4 . The dispersion curves depicted in Fig. 4 agree with [17] as expected.

Expressions (19) yield $\Delta_\alpha = \alpha^U(\Delta' - 2)$ and $\Delta_\beta = \beta^U \Delta'$ with Δ' defined as

$$\Delta' = \tanh(d^U k^U)/k^U + d^U/\cosh^2(d^U k^U)\,. \tag{47a}$$

Expressions (21) and (42) show that, at a dispersion curve $\Delta = 0$, one has

$$-\Delta_k = \tanh(d^U k^U) - d^U k^U/\cosh^2(d^U k^U)\,. \tag{47b}$$

Thus, one has $\Delta_k < 0$, and (17) yields $\text{sign}\Delta_f = \text{sign}\alpha$. The parametric equations (37) for the farfield wave patterns then become

$$\begin{Bmatrix} x_n^U \\ y_n^U \end{Bmatrix} = \frac{2\,n\,\pi}{-\Delta_k\,k^U} \begin{Bmatrix} -|\alpha^U|\,(2-\Delta') \\ \text{sign}\alpha^U\,\beta^U\,\Delta' \end{Bmatrix} \quad \text{with} \begin{Bmatrix} x_n^U = x_n/F^2 = X_n\,g/\mathcal{U}^2 \\ y_n^U = y_n/F^2 = Y_n\,g/\mathcal{U}^2 \end{Bmatrix}. \tag{48}$$

Expressions (48), (47) and (45) provide parametric equations that define the farfield wave patterns generated by a ship advancing at constant speed in calm water of uniform finite depth. Expressions (48), (47) and (46a) yield the deep-water approximation

$$\begin{Bmatrix} x_n^U \\ y_n^U \end{Bmatrix} \sim \frac{2\,n\,\pi}{k^U} \begin{Bmatrix} -\sqrt{k^U}\,(2-1/k^U) \\ \pm\sqrt{1-1/k^U} \end{Bmatrix} \quad \text{with } 1 \le k^U \quad \text{as } d^U \to \infty\,. \tag{49}$$

The wave patterns defined by the parametric equations (48) are depicted in the upper half of Fig. 5 for $d^U = \infty$ (left), 1.05 (center) and 1.01 (right), and in the lower half of Fig. 5 for $d^U = 0.995$ (right), 0.7 (center) and 0.3 (left). The wave patterns shown in Fig. 5 agree with [18,19] as expected.

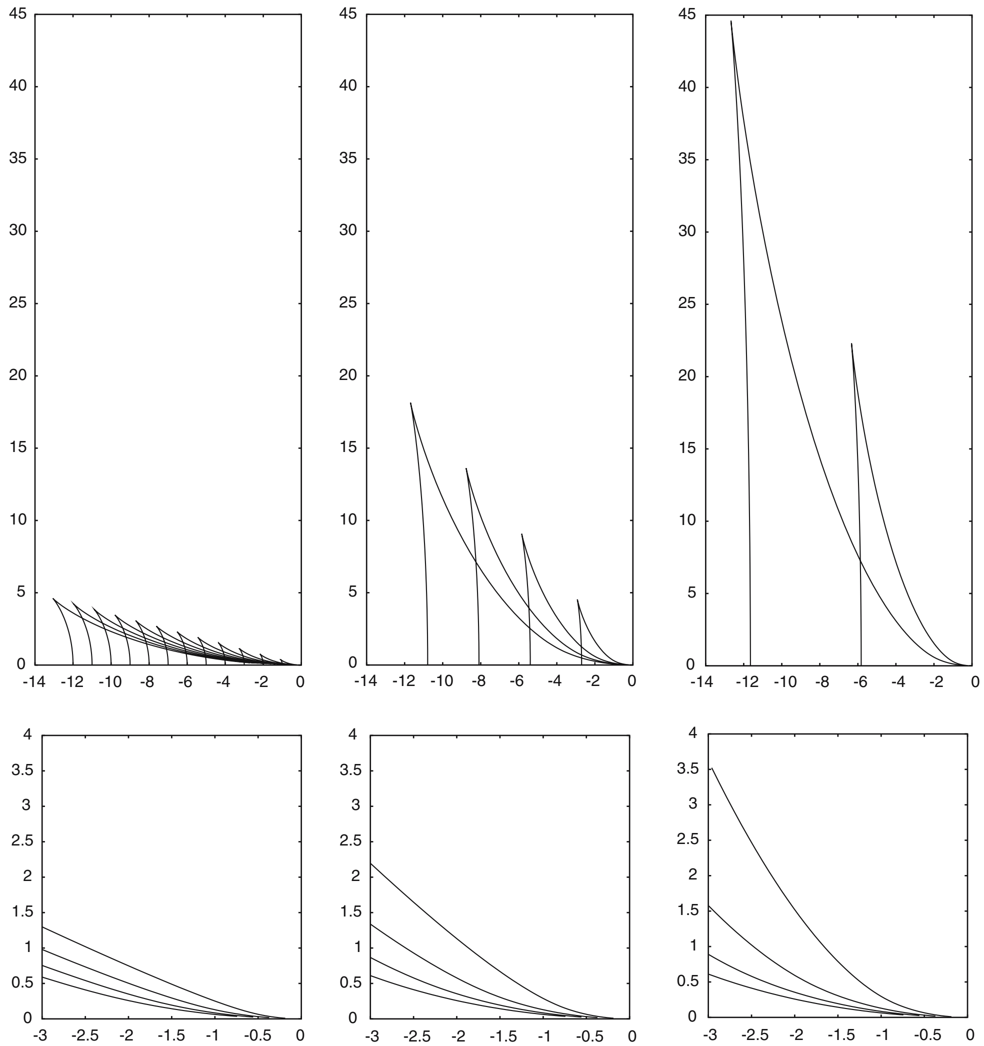

Fig. 5 Steady ship waves in finite water depth: farfield wave patterns for (top row) $d^U = \infty$ (left column), 1.05 (center) and 1.01 (right) and for (lower row) $d^U = 0.995$ (right column), 0.7 (center) and 0.3 (left)

Expressions (48), (47) and (46b) yield

$$\begin{Bmatrix} x_n^U \\ y_n^U \end{Bmatrix} \sim \frac{-6\,n\,\pi}{(d^U)^{5/2}(k^U)^3} \begin{Bmatrix} 1-d^U+2\,(d^U)^3(k^U)^2/3 \\ \pm\sqrt{d^U}\sqrt{1-d^U+(d^U)^3(k^U)^2/3} \end{Bmatrix} \text{ as } k^U \to 0\,. \tag{50}$$

Expressions (50), where $0 \le d^U \le 1$, show that one has $\sqrt{\left(x_n^U\right)^2+\left(y_n^U\right)^2} \to \infty$ as $k^U \to 0$. Thus, the wave pattern for $0 \le d^U \le 1$ has an asymptote defined by $y_n^U/x_n^U \sim \pm\sqrt{d^U}/\sqrt{1-d^U}$. The angle between this asymptote and the x-axis is then given by

$$\theta_a = \pm\sin^{-1}(\sqrt{d^U}) \quad \text{with } 0 \le d^U \le 1\,, \tag{51}$$

in agreement with [18, 19]. Thus, one has $\theta_a \to 0$ as $d^U \to 0$ and $\theta_a \to \pm 90°$ as $d^U \to 1$.

Figure 4 shows that the dispersion curves for $1 < d^U$ have an inflection point. Expressions (19) yield

$$\Delta_\alpha = T\alpha/k - 2\,\alpha \quad \Delta_\beta = T\beta/k \quad \text{with } T = \tanh k^d + k^d/\cosh^2 k^d \text{ and } k^d = kd\,.$$

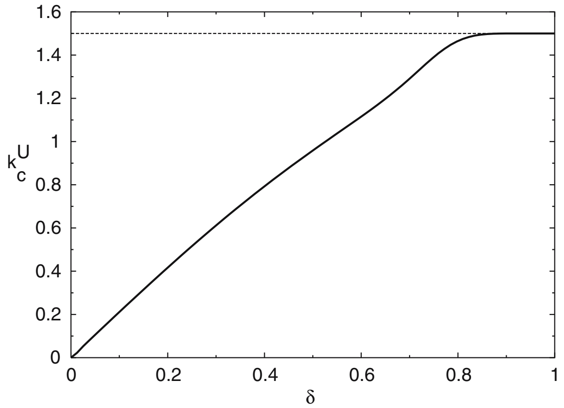

Fig. 6 Steady ship waves in finite water depth: variation of root $k_c^U(d^U)$ of cusp-line equation (52) in terms of water depth $0 \le \delta = \sqrt{1-1/d^U} \le 1$ with $1 \le d^U$

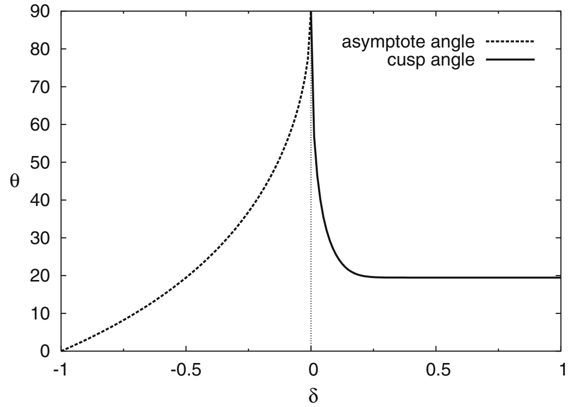

Fig. 7 Steady ship waves in finite water depth: variation of asymptote-line angle θ_a (dashed line) and cusp-line angle θ_c (solid line) in terms of water depth $-1 \le \delta = (d^U-1)/(d^U+1) \le 1$. Left half $-1 \le \delta \le 0$ and right half $0 \le \delta \le 1$ correspond to $0 \le d^U \le 1$ and $1 \le d^U$, and depict asymptote angle θ_a and cusp angle θ_c, respectively

Here, d, k, α and β stand for the speed-scaled variables d^U, k^U, α^U and β^U (the superscript U is momentarily ignored for shortness). One then has

$$k\Delta_{\alpha\alpha} = \frac{\beta^2 T - \alpha^2 T'}{k^2} - 2\,k \quad k\Delta_{\beta\beta} = \frac{\alpha^2 T - \beta^2 T'}{k^2} \qquad -k\Delta_{\alpha\beta} = \frac{\alpha\beta}{k^2}\,(T+T')$$

with $T' = 2\,k^d(k^d \tanh k^d - 1)/\cosh^2 k^d$. Equation 41 and the relation $\beta^2 = k^2 - \alpha^2$ then yield

$$T^2\left(T - 2\,\frac{\alpha^2}{k}\right) + 4\,\frac{\alpha^4}{k^2}\,(T+T') - 2\,k\left(T^2 + 2\,\frac{\alpha^2}{k}\,T'\right) = 0\,.$$

The dispersion relation (42) yields $\alpha^2/k = \tanh k^d$. One then obtains

$$(T^2 + 2\,T' \tanh k^d)\,k_c^U = 2\,(T+T')\tanh^2 k^d - T^2(\tanh k^d - T/2) \text{ with} \tag{52a}$$

$$T = \tanh k^d + k^d/\cosh^2 k^d \quad T' = 2\,k^d(k^d \tanh k^d - 1)/\cosh^2 k^d \quad k^d = d^U k_c^U. \tag{52b}$$

In the deep-water limit $d^U \to \infty$, (52b) yields $\tanh k^d \sim 1$, $T \sim 1$ and $T' \sim 0$. Expressions (52a), (46a) and (49) then yield

$$k_c^U \to \frac{3}{2} \qquad \alpha_c^U \to \pm\sqrt{\frac{3}{2}} \qquad \beta_c^U \to \pm\,\frac{\sqrt{3}}{2} \qquad \frac{y_c^U}{x_c^U} \to \frac{\pm 1}{2\sqrt{2}} \qquad \text{as } d^U \to \infty$$

and the angle between the cusp lines and the x-axis is given by $\theta_c \to \pm 19°28'$ as $d^U \to \infty$, in agreement with Kelvin's classical result. Equations 52 and 50 can be verified to yield

$$k_c^U \to 0 \quad y_c^U/x_c^U \to \pm\infty \quad \text{as } d^U \to 1\,.$$

Thus, the angle θ_c between the cusp lines and the x axis is given by $k_c^U \to 90°$ as $d^U \to 1$, as well known; e.g. [18, 19]. The initial approximation $k_c^U = (3/2)\tanh\sqrt{d^U - 1}$ can be used to determine the root $k_c^U(d^U)$ of (52). This root is depicted in Fig. 6. The angle θ_c between the cusp lines and the x-axis is then defined by (48), (45a) and (47a) as

$$\tan\theta_c = \frac{y_c^U}{x_c^U} = \frac{\pm\,k_c^U\sqrt{1 - \tanh(d^U k_c^U)/k_c^U\,\Delta'}}{\sqrt{k_c^U\,\tanh(d^U k_c^U)}\,(2 - \Delta')} \quad \text{with } \Delta' = \frac{\tanh\left(d^U k_c^U\right)}{k_c^U} + \frac{d^U}{\cosh^2\left(d^U k_c^U\right)}\,. \tag{53}$$

The asymptote-lines angle $0 \le \theta_a \le 90°$ defined by (51) and the cusp-lines angle $19°28' \le \theta_c \le 90°$ defined by (53) with (52) are depicted in Fig. 7.

6 Elementary nearfield waves

Expressions (25) and (9) define a Fourier superposition of elementary waves

$$\mathcal{R}^F W = [\,1+\operatorname{sign}\Delta_f \operatorname{sign}\Delta_k \operatorname{sign}(\alpha\, x+\beta\, y)\,]\, \mathrm{e}^{\,\mathrm{i}\,(\alpha\, x+\beta\, y)} \cosh[\,k\,(z+d)\,]/\cosh(k\,d)\,, \tag{54}$$

where the Fourier variables α and β lie on the dispersion curves $\Delta(\alpha\,,\beta\,;f,F,d)=0$ related to the dispersion function (10). A radiation condition and the boundary condition (4b) at the sea floor are satisfied by the elementary wave (54). Expression (54) would also satisfy the Laplace equation (4a) and the boundary condition (4c) at the free surface if not for the sign function $\operatorname{sign}(\alpha\, x+\beta\, y)$. This sign function is inconsequential in the farfield because the dominant contributions to the Fourier integral (25a) stem from points of stationary phase, where the relation (29) holds and one has $\mathcal{R}^F=2$. Thus, for all practical purposes, the flow defined by the Fourier representation (25) satisfies the Laplace equation and the free-surface condition in the farfield. However, equation (25) does not satisfy the Laplace equation (and the free-surface condition) in the nearfield. The fact that (25) does not satisfy the Laplace equation in the nearfield is a major limitation for nearfield flows. Specifically, this restriction precludes the use of (25) as a nearfield-flow representation; e.g. (25) cannot be used to construct Green functions or in a spectral representation of nearfield flows. However, the elementary wave (54) can be modified to satisfy the Laplace equation in the nearfield, as shown below.

The deep-water limit $d\to\infty$ is considered first. Consider the function

$$\mathcal{R}^* = 2\Big/\left[1+\exp\left(-\mu\,\frac{\alpha\, x+\beta\, y-\mathrm{i}\,k\,z}{\sigma}\right)\right], \tag{55}$$

where σ is a positive real function of α and β. One has

$$\mathcal{R}^* \sim 1+\operatorname{sign}\mu\ \operatorname{sign}(\alpha\, x+\beta\, y) \quad \text{as}\ |\,\alpha\, x+\beta\, y\,|/\sigma\to\infty\,.$$

This farfield approximation shows that the radiation function $\mathcal{R}^F$ in (54) may be replaced by the function $\mathcal{R}^*$ if μ in (55) is chosen as in (22a). Thus, a radiation condition is (effectively; in the manner explained earlier) satisfied by the elementary wave

$$W^*(\mathbf{x}) = 2\,\mathrm{e}^{\,\mathrm{i}\,(\alpha\, x+\beta\, y)+k\,z}\Big/\left[1+\exp\left(-\mu\,\frac{\alpha\, x+\beta\, y-\mathrm{i}\,k\,z}{\sigma}\right)\right], \tag{56}$$

where μ is given by (22a) and σ is a positive real function of α and β. Furthermore, the elementary wave W^* satisfies the deep-water boundary condition $W^*\to 0$ as $z\to-\infty$, and can be verified to satisfy the Laplace equation if $k=\sqrt{\alpha^2+\beta^2}$. The elementary wave W^* also satisfies the free-surface boundary condition (4c) in the farfield if the Fourier variables α and β satisfy the dispersion relation $\Delta=0$, with Δ given by (11). The elementary deep-water wave W^* can be expressed as

$$W^*(\mathbf{x}) = \mathrm{e}^{\,\mathrm{i}\,(\alpha\, x+\beta\, y)+k\,z}\left(1+\mu\tanh\frac{\alpha\, x+\beta\, y-\mathrm{i}\,k\,z}{2\,\sigma}\right).$$

Here, the relation $\tanh(\mu\mathcal{Z})=\mu\tanh(\mathcal{Z})$ for $\mu=\pm1$ was used. Thus, one has

$$W^*(\mathbf{x}) = \mathrm{e}^{\,\mathrm{i}\,(\alpha\, x+\beta\, y)+k\,z}\left(1+\mu\,\frac{\sinh(\varphi/\sigma)-\mathrm{i}\sin(V/\sigma)}{\cosh(\varphi/\sigma)+\cos(V/\sigma)}\right) \tag{57}$$

with $\varphi=\alpha\, x+\beta\, y$ and $V=k\,z$.

In uniform finite water depth d, the elementary wave (56) becomes

$$W^*(\mathbf{x}) = \mathrm{e}^{\,\mathrm{i}\,(\alpha x+\beta y)}\,\frac{\Lambda^+ + \Lambda^-}{\cosh(k\,d)} \quad \text{with } \Lambda^\pm = \frac{\exp[\,\pm k\,(z+d)\,]}{1+\exp[-\mu\,\{\,\alpha\,x+\beta\,y \mp \mathrm{i}\,k\,(z+d)\}/\sigma\,]}\,,$$

where $k = \sqrt{\alpha^2+\beta^2}$ and μ is given by (22a). This elementary wave satisfies a radiation condition, the Laplace equation and the boundary condition $W_z^* = 0$ at the sea floor $z = -d$, and can be expressed as

$$W^*(\mathbf{x}) = \mathrm{e}^{\,\mathrm{i}\,(\,\alpha x+\beta y\,)}\,\frac{\cosh\,[k\,(z+d)]}{\cosh(k\,d)}\left(1+\mu\,\frac{\sinh(\varphi/\sigma) - \mathrm{i}\tanh V\sin(V/\sigma)}{\cosh(\varphi/\sigma)+\cos(V/\sigma)}\right). \tag{58}$$

Here, $V = k\,(z+d)$. The elementary wave (58) can be expressed as $\mathcal{R}^N W$, where W is the elementary wave (9) and the function $\mathcal{R}^N$, called nearfield radiation function here, is defined as

$$\mathcal{R}^N = 1+\mathrm{sign}\Delta_f\,\mathrm{sign}\Delta_k\,\Theta\,. \tag{59a}$$

Here, (22a) was used and Θ stands for the function

$$\Theta(\mathbf{x}) = \frac{\tanh(\varphi/\sigma) - \mathrm{i}\,\tanh V\sin(V/\sigma)\,/\cosh(\varphi/\sigma)}{1+\cos(V/\sigma)\,/\cosh(\varphi/\sigma)} \quad \text{with } \left\{\begin{matrix}\varphi = \alpha\,x+\beta\,y \\ V = k\,(z+d\,)\end{matrix}\right\}. \tag{59b}$$

In the deep-water limit $d \to \infty$, one has $\tanh V \sim 1$ and $V = k\,z$ in (59b). Expression (59b) yields

$$\Theta \sim \tanh(\varphi/\sigma) \sim \mathrm{sign}(\alpha\,x+\beta\,y) \quad \text{as} \quad h = \sqrt{x^2+y^2} \to \infty\,. \tag{59c}$$

The farfield approximation (59c) shows that the nearfield radiation function $\mathcal{R}^N$ defined by (59a) and (59b) is asymptotically equivalent to the farfield radiation function $\mathcal{R}^F$ given by (25b) in the farfield limit $h \to \infty$.

Water waves are significant only in the layer $-z < d^\infty$ with $k\,d^\infty = \pi$, and the water depth d is effectively infinite for $d^\infty < d$. Expression (59b) and the relations $1 \le \cosh(\varphi/\sigma)$ and $-1 \le \cos(V/\sigma) \le 1$ show that the function Θ is finite except if $\varphi = 0$ and $\cos(V/\sigma) = -1$. Expression (59b) for V then shows that Θ is finite if

$$k\,d/\pi = (d/d^\infty)\,k\,d^\infty/\pi < \sigma. \tag{60a}$$

In deep water, for which one has $V = k\,z$, Θ is finite if

$$-\,k\,z/\pi = (-z/d^\infty)\,k\,d^\infty/\pi < \sigma. \tag{60b}$$

The conditions (60) show that the function Θ given by (59b) is finite within the flow domain where waves are significant if $1 < \sigma$. The variation of the function φ/σ that corresponds to a half-period of the trigonometric function $\mathrm{e}^{\,\mathrm{i}\,\varphi}$ is given by π/σ. Thus, the choice $\sigma = 1$ implies that the functions $\tanh(\varphi/\sigma)$ and Θ become practically equal to ± 1 within half a period (wavelength) of the function $\mathrm{e}^{\,\mathrm{i}\,\varphi}$, i.e., that "nearfield effects" are approximately confined to half a wavelength.

The waves defined by the Fourier representation

$$\sum_{\Delta=0} \int_{\Delta=0} \mathrm{d}s\,A\,\mathcal{R}^N W, \tag{61}$$

where W is the elementary wave (9), $\mathcal{R}^N$ is the nearfield radiation function (59), and A is an amplitude function (determined by a nearfield body boundary condition), satisfy a radiation condition, the Laplace equation and the sea-floor boundary condition. This Fourier superposition of elementary "nearfield waves" also satisfies the free-surface boundary condition in the farfield. However, the free-surface condition is only approximately satisfied in the nearfield, i.e., in the vicinity of a wave generator such as a ship or offshore structure. The nearfield region is no larger than half a wavelength if $1 < \sigma$ in (59b).

7 Conclusions

The one-dimensional Fourier superposition

$$\sum_{\Delta=0} \int_{\Delta=0} \mathrm{d}s\, a\, W \tag{62a}$$

of classical elementary waves

$$W(\mathbf{x}\,;\alpha\,,\beta) = \mathrm{e}^{\,\mathrm{i}\,(\alpha\,x+\beta\,y)} \cosh\,[\,k\,(z+d)\,]/\cosh(k\,d) \quad \text{with} \quad k=\sqrt{\alpha^2+\beta^2} \tag{62b}$$

of amplitude a, along the dispersion curves defined by the dispersion relation

$$\Delta(\alpha,\beta;f,F,d) = k\tanh(k\,d) - (f-F\alpha)^2 = 0\,, \tag{62c}$$

satisfies the Laplace equation and the boundary conditions at the sea floor $z=-d$ and the free surface $z=0$. However, this classical Fourier superposition of elementary waves, associated with a time-harmonic flow defined by a potential

$$\mathfrak{Re}\;\phi(\mathbf{x})\,\mathrm{e}^{\,\mathrm{i}\,f\,t} \quad \text{with} \quad f=\omega\sqrt{L/g}\,, \tag{62d}$$

is not a satisfactory representation of farfield waves. For instance, expressions (62) do not preclude steady ship waves ahead of a ship advancing in calm water. In fact, it is well known that unique time-harmonic flows of the form (62d) can only be obtained by invoking a radiation condition, a critical farfield condition that specifies that wave energy is radiated away from the wave generator [3–7].

In this study, the modified elementary waves $\mathcal{R}^F W$ and the related Fourier superposition

$$\sum_{\Delta=0} \int_{\Delta=0} \mathrm{d}s\, A\, \mathcal{R}^F W \quad \text{with } \mathcal{R}^F = 1+\operatorname{sign}\Delta_f\,\operatorname{sign}\Delta_k\,\operatorname{sign}(\alpha\,x+\beta\,y) \tag{63a}$$

have been obtained using Lighthill's approach, which considers flows that slowly grow from rest at time $t=-\infty$, in accordance with a potential

$$\mathfrak{Re}\;\phi(\mathbf{x})\,\mathrm{e}^{\,\varepsilon\,t+\mathrm{i}\,f\,t} = \mathfrak{Re}\;\phi(\mathbf{x})\,\mathrm{e}^{\,\mathrm{i}\,(f-\mathrm{i}\,\varepsilon)\,t} \quad \text{with} \;\; 0<\varepsilon\ll 1\,. \tag{63b}$$

The modified elementary waves $\mathcal{R}^F W$ have been obtained—in a simple way, using only elementary considerations—as solutions of the farfield boundary-value problem (8) associated with (63b). Unlike the potential (62d), which contains no information about initial conditions, initial conditions are embedded in (63b). The radiation function $\mathcal{R}^F$ in (63a), associated with the potential (63b) and the related dispersion relation

$$\Delta(\alpha+\mathrm{i}\,\varepsilon\,\alpha_1,\beta+\mathrm{i}\,\varepsilon\,\beta_1\,;f-\mathrm{i}\,\varepsilon\,,F,d) = 0 \tag{63c}$$

introduces restrictions that do not exist in the classical representation (62). Specifically, the function $\mathcal{R}^F$ defined by (63a) divides the dispersion curves $\Delta=0$ into "dead" and "alive" sections (that depend on x and y), where one has $\mathcal{R}^F=0$ or $\mathcal{R}^F=2$. These restrictions, introduced via the function $\mathcal{R}^F$, account for the information that is included in the potential (63b) but is lost if the time-initialization parameter ε is set null in (63b) and (63c). Thus, Lighthill's approach, associated with the potential (63b), circumvents the need for a radiation condition. Alternatively, one may consider that a radiation condition is effectively satisfied ab initio (with no need for additional consideration) by the modified elementary waves $\mathcal{R}^F W$ and the related Fourier superposition (63a). Accordingly, the function $\mathcal{R}^F$ is called farfield radiation function here.

The modified elementary wave solution $\mathcal{R}^F W$ is given by the product of the classical elementary time-harmonic water wave W by the radiation function $\mathcal{R}^F$. This function is defined explicitly in terms of the dispersion function Δ. Indeed, the modification $\mathcal{R}^F W$ of the elementary wave W can readily be applied to a wide class of dispersive waves. Specifically, elementary waves W associated with a given dispersive medium, characterized by a dispersion function Δ, can easily be made to (indirectly) satisfy a radiation condition via multiplication of W by the farfield radiation function $\mathcal{R}^F$ given by (63a), which is defined explicitly in terms of the derivative Δ_f of the dispersion function Δ with respect to the frequency f and the derivative Δ_k of Δ in the radial direction $(\alpha,\beta)/k$. Thus, the Fourier superposition (63a) of modified elementary waves $\mathcal{R}^F W$ is readily applicable to a wide class of dispersive waves. Indeed, farfield dispersive waves may be expressed in the form (63a) ab initio.

The Fourier superposition (63a) of modified elementary waves $\mathcal{R}^F W$ satisfies a radiation condition, as already noted, and thus may be expected to yield correct farfield waves, notably wave patterns. For purposes of illustration and verification, farfield waves have been considered here using asymptotic analysis, based on Kelvin's method of stationary phase, of the Fourier integral (63a). In fact, only elementary stationary-phase considerations were used to determine the main properties of farfield waves (phase and group velocities, wave patterns, asymptote and cusp lines), and to illustrate the fundamental relationships that exist between farfield waves in the physical plane $(x\,,y)$ and the dispersion function $\Delta(\alpha\,,\beta\,;f,F,d)$ and related dispersion curves $\Delta=0$ in the Fourier plane $(\alpha\,,\beta)$. Expressions (13), (30c) and (38) for the group velocity, the relations (30a) and (30b) between farfield waves in the physical plane $(x\,,y)$ and the dispersion curves $\Delta=0$ in the Fourier plane, the parametric equations (37) of farfield wave patterns, expression (40) to determine asymptote lines of a farfield wave pattern, and the property that cusp lines of farfield wave patterns are determined by inflection points of dispersion curves, only involve the dispersion function Δ and thus are valid for a broad class of dispersive waves. These relations associated with farfield waves agree with, or are modifications of, similar relations given in the literature, e.g. [2,11,14–16]. In particular, detailed illustrations of the relationship between the dispersion function and the related dispersion curves $\Delta=0$ in the Fourier plane $(\alpha\,,\beta)$ and the corresponding farfield waves in the physical plane $(x\,,y)$ are given in [14,15] for diffraction-radiation by a ship advancing through regular waves in deep water. A verification of the foregoing "farfield-waves" relations, and of the underlying Fourier representation (63a) of modified elementary waves $\mathcal{R}^F W$, has been given here by considering two illustrative applications to water waves in uniform finite water depth: diffraction-radiation of time-harmonic waves by an offshore structure, and steady ship waves.

The elementary farfield wave $\mathcal{R}^F W$ would satisfy the Laplace equation and the boundary condition at the free surface if not for the sign function $\text{sign}(\alpha\,x+\beta\,y)$. This sign function is inconsequential in the farfield because the dominant contributions to the Fourier integral (63a) stem from points of stationary phase, where the relation (29) holds and one has $\mathcal{R}^F=2$. Thus, for all practical purposes, the flow defined by the Fourier representation (63a) satisfies the Laplace equation and the free-surface condition in the farfield. However, (63a) does not satisfy the Laplace equation (and the free-surface condition) in the nearfield. The fact that (63a) does not satisfy the Laplace equation in the nearfield is a major limitation for nearfield flows. Specifically, this restriction precludes the use of (63a) as a nearfield-flow representation; e.g. (63a) cannot be used to construct Green functions or in a spectral representation. However, the elementary farfield waves $\mathcal{R}^F W$ in the Fourier superposition (63a) can be replaced by nearfield waves $\mathcal{R}^N W$ that satisfy the Laplace equation everywhere (in the nearfield as well as the farfield). Thus, (63a) becomes

$$\sum_{\Delta=0}\int_{\Delta=0} ds\, A\,\mathcal{R}^N\,W \quad \text{with} \;\; \mathcal{R}^N=1+\text{sign}\Delta_f\,\text{sign}\Delta_k\,\Theta\,. \tag{64}$$

Here, the function Θ is given by (59b). One has $\mathcal{R}^N\sim\mathcal{R}^F$ in the farfield, in accordance with (59c). The function $\mathcal{R}^N$ in (64) is called nearfield radiation function. The Fourier superposition (64) of elementary nearfield waves $\mathcal{R}^N W$ satisfies the Laplace equation, a radiation condition (effectively), the boundary condition at the sea floor, and also satisfies the free-surface boundary condition in the farfield. The Fourier superposition (64) of elementary waves can then be used to represent nearfield flows. In particular, the elementary nearfield waves can be used in a spectral representation of nearfield flows. The elementary nearfield waves defined by (64) and (62b) can also be used to obtain simple free-surface Green functions. Specifically, Green functions that satisfy the free-surface boundary condition accurately in the farfield and approximately (to leading order) in the nearfield can easily be constructed using the nearfield waves $\mathcal{R}^N W$, in the manner shown in [13]. Indeed, the wave components in the Green functions for time-harmonic and steady free-surface flows about ships or offshore structures given in [13] are precisely of the form (64), as one would expect. The practical usefulness of these simple Green functions, and hence of the underlying elementary nearfield waves $\mathcal{R}^N W$ given here, is demonstrated in [8]. Specifically, reference [8] shows that the simple Green functions given in [13], which only satisfy the free-surface boundary condition to leading order in the nearfield, are identical to the classical free-surface Green functions (which satisfy the linearized free-surface condition everywhere) in the farfield as expected, and that nearfield differences are relatively small.

Expressions (63a), (64) and (59b) correspond to time-harmonic flows defined by potentials of the form (62d) and elementary waves (62b). Expressions (63a), (64) and (59b) can readily be applied, with the elementary modifications given in Appendix 6, to time-harmonic flows and/or elementary waves defined by the alternative expressions

$$\mathfrak{Re}\;\phi(\mathbf{x})\,\mathrm{e}^{-\mathrm{i}ft} \qquad W(\mathbf{x};\alpha,\beta)=\mathrm{e}^{-\mathrm{i}(\alpha x+\beta y)}\cosh[\,k\,(z+d)\,]/\cosh(k\,d)$$

used in the literature.

In summary, the main results and contribution of this study are two new classes of elementary (time-harmonic) water waves (observed from a Galilean frame of reference) that satisfy a radiation condition ab initio. These two related classes of elementary waves, called elementary farfield and nearfield waves, are simple modifications of classical elementary waves. In particular, the elementary farfield wave is given by the product of the classical elementary wave by a farfield radiation function that is explicitly defined in terms of the dispersion function. This radiation function is valid not only for water waves, but more generally for a broad class of dispersive waves. The nearfield waves given here are useful to represent nearfield flows, via a spectral representation or a Green function. The elementary farfield and nearfield waves have been obtained in a remarkably simple manner, using only fundamental considerations and elementary analysis, by considering flows that slowly start from rest at $t=-\infty$, in accordance with Lighthill's approach.

Appendix 1

Let k^ω be expressed in the form $k^\omega = P/\sqrt{d^\omega}$. Thus, (27a) becomes $\sqrt{d^\omega}/P = \tanh(\sqrt{d^\omega}P)$. Use of the Taylor series for $\tanh(d^\omega k^\omega)$ then yields $1=\mu - d^\omega\mu^2/3+2\,(d^\omega)^2\mu^3/15+\cdots$ with $\mu = P^2$. Substitution of the series $\mu\approx 1+C_1\,d^\omega+C_2\,(d^\omega)^2+\cdots$ yields $C_1=1/3$ and $C_2=4/45$. One then has $k^\omega = P/\sqrt{d^\omega}=\sqrt{\mu}\,/\sqrt{d^\omega}\sim[\,1+d^\omega/3+4\,(d^\omega)^2/45\,]^{1/2}/\sqrt{d^\omega}\sim[\,1+d^\omega/6+11\,(d^\omega)^2/360\,]/\sqrt{d^\omega}$.

Appendix 2

The wave-pattern (37) can be expressed as

$$\frac{(x_n\,,y_n)}{2\,n\,\pi}=\mu\,\frac{(\Delta_\alpha\,,\Delta_\beta)}{\alpha\,\Delta_\alpha+\beta\,\Delta_\beta}\quad\text{ with } \mu=\mathrm{sign}\Delta_f\,\mathrm{sign}\Delta_k\,.$$

Consider two points $(\alpha\,,\beta)$ and $(\alpha+\mathrm{d}\alpha\,,\beta+\mathrm{d}\beta)$ of a dispersion curve $\Delta(\alpha\,,\beta\,;f,F,d)=0$. This relation yields $\Delta_\alpha\,\mathrm{d}\alpha+\Delta_\beta\,\mathrm{d}\beta=0$; i.e. one has $\mathrm{d}\alpha=\sigma\,\Delta_\beta\,\mathrm{d}s$ and $\mathrm{d}\beta=-\sigma\,\Delta_\alpha\,\mathrm{d}s$ with $\mathrm{d}s=\sqrt{\mathrm{d}\alpha^2+\mathrm{d}\beta^2}$ and $\sigma=\pm1$. The wave-front point $(x_n\,,y_n)$ that corresponds to the dispersion-curve point $(\alpha\,,\beta)$ becomes $(x_n+\mathrm{d}x_n\,,y_n+\mathrm{d}y_n)$ with

$$\frac{\mathrm{d}x_n/\mathrm{d}s}{2\,n\,\pi\,\mu\,\sigma}=\Delta_\beta\,\frac{\partial}{\partial\alpha}\,\frac{\Delta_\alpha}{\alpha\,\Delta_\alpha+\beta\,\Delta_\beta}-\Delta_\alpha\,\frac{\partial}{\partial\beta}\,\frac{\Delta_\alpha}{\alpha\,\Delta_\alpha+\beta\,\Delta_\beta}=\frac{\beta\,N}{(\alpha\,\Delta_\alpha+\beta\,\Delta_\beta)^2}\,,$$

$$\frac{\mathrm{d}y_n/\mathrm{d}s}{2n\pi\,\mu\,\sigma} = \Delta_\beta\,\frac{\partial}{\partial\alpha}\,\frac{\Delta_\beta}{\alpha\,\Delta_\alpha+\beta\,\Delta_\beta} - \Delta_\alpha\,\frac{\partial}{\partial\beta}\,\frac{\Delta_\beta}{\alpha\,\Delta_\alpha+\beta\,\Delta_\beta} = \frac{-\alpha\,N}{(\alpha\,\Delta_\alpha+\beta\,\Delta_\beta)^2}$$

with $N = \Delta_\beta^2\,\Delta_{\alpha\alpha} - 2\,\Delta_\alpha\,\Delta_\beta\,\Delta_{\alpha\beta} + \Delta_\alpha^2\,\Delta_{\beta\beta}$. Expression (12) then yields $v_p^x\,\mathrm{d}x_n + v_p^y\,\mathrm{d}y_n = 0$. Thus, the phase velocity $\mathbf{v}_\mathrm{p}$ is orthogonal to the constant-phase curves (e.g. wave crests and troughs) defined by (37), as expected.

Appendix 3

Diffraction-radiation by a ship advancing through regular waves in deep water is now considered for purposes of illustration. The dispersion function (11) defines three distinct dispersion curves if $\tau < 1/4$, or two curves if $1/4 < \tau$; see e.g. [13,14]. The three dispersion curves for $\tau < 1/4$ have no "asymptote point", i.e., no point where $\Delta_k = 0$. However, one of the two dispersion curves for $1/4 < \tau$ has an asymptote point, which is defined by

$$\alpha_a = -f/F \quad \beta_a = \pm\sqrt{16\tau^2-1}\,f/F \quad k_a = 4f^2,$$

as can be easily verified [13]. The dispersion curve that contains the asymptote point (α_a, β_a) can be broken into two joint, but separate, curves. In this approach, used in [13], one then has three dispersion curves, for both $\tau < 1/4$ and $1/4 < \tau$, within which the functions Δ_k and Δ_f in (25b) have constant signs, given by (36) in [13]. The dispersion curves defined by the dispersion function (10) for diffraction-radiation by a ship in finite water depth can similarly be broken into three dispersion curves, for every value of τ, within which the signs of Δ_k and Δ_f remain constant [20].

Appendix 4

The unit vector $(\Delta_\alpha, \Delta_\beta)/\|\nabla\Delta\|$ is normal to a dispersion curve $\Delta(\alpha,\beta) = 0$, as already noted. Thus, the unit vector $(\Delta_\beta, -\Delta_\alpha)/\|\nabla\Delta\|$ is tangent to a dispersion curve, and the derivative $\mathrm{d}/\mathrm{d}s$ in a direction tangent to a dispersion curve is given by

$$\frac{(\Delta_\beta, -\Delta_\alpha)\cdot(\partial_\alpha, \partial_\beta)}{\|\nabla\Delta\|} = \frac{\Delta_\beta}{\|\nabla\Delta\|}\,\frac{\partial}{\partial\alpha} - \frac{\Delta_\alpha}{\|\nabla\Delta\|}\,\frac{\partial}{\partial\beta}$$

An inflection point on a dispersion curve is determined by the condition

$$0 = \frac{\mathrm{d}\tan\delta}{\mathrm{d}s} = \frac{\mathrm{d}\,(\Delta_\beta/\Delta_\alpha)}{\mathrm{d}s} \quad \text{i.e.,} \quad \Delta_\beta\,\frac{\partial}{\partial\alpha}\left(\frac{\Delta_\beta}{\Delta_\alpha}\right) - \Delta_\alpha\,\frac{\partial}{\partial\beta}\left(\frac{\Delta_\beta}{\Delta_\alpha}\right) = 0\,.$$

This condition yields

$$\Delta_\beta^2\,\Delta_{\alpha\alpha} - 2\Delta_\alpha\,\Delta_\beta\,\Delta_{\alpha\beta} + \Delta_\alpha^2\,\Delta_{\beta\beta} = 0. \tag{65a}$$

This equation determines the inflection point(s) of a dispersion curve that is defined by an implicit equation $\Delta(\alpha,\beta) = 0$. Similarly, an inflection point of a dispersion curve defined by the parametric equations $\alpha = A(t)$ and $\beta = B(t)$ is determined by the equation

$$\frac{\mathrm{d}A}{\mathrm{d}t}\,\frac{\mathrm{d}^2B}{\mathrm{d}t^2} - \frac{\mathrm{d}B}{\mathrm{d}t}\,\frac{\mathrm{d}^2A}{\mathrm{d}t^2} = 0\,. \tag{65b}$$

For a dispersion curve defined by the polar equation $k = K(\gamma)$, Eq. 65b yields

$$K^2 + 2\,(\mathrm{d}K/\mathrm{d}\gamma)^2 - K\,\mathrm{d}^2K/\mathrm{d}\gamma^2 = 0\,. \tag{65c}$$

Finally, for a dispersion curve defined by the explicit equations $\beta = B(\alpha)$ or $\alpha = A(\beta)$ Eq. 65b becomes

$$\mathrm{d}^2B/\mathrm{d}\alpha^2 = 0 \quad \text{or} \quad \mathrm{d}^2A/\mathrm{d}\beta^2 = 0\,, \tag{65d}$$

respectively. The alternative equation (65) can be used to determine the inflection point(s) of a dispersion curve, depending on the manner in which the dispersion curve is defined.

Appendix 5

Use of the Taylor series for $\tanh(d^{U\gamma} k^{U\gamma})$ in (43) yields

$3\,(1-1/d^{U\gamma}) - (d^{U\gamma} k^{U\gamma})^2 + 2\,(d^{U\gamma} k^{U\gamma})^4/5 - 17\,(d^{U\gamma} k^{U\gamma})^6/315 = 0\,.$

The change of variable $d^{U\gamma} k^{U\gamma} = \sqrt{3\,\eta\,P}$ with $1-1/d^{U\gamma} = \eta$ yields $1 - P + 6\,\eta\,P^2/5 - 51\,\eta^2 P^3/35 = 0$. Let $P \sim 1 + C_1\,\eta + C_2\,\eta^2$, which yields $P^2 \sim 1 + 2\,C_1\,\eta$ and $P^3 \sim 1$. One then obtains $C_1 = 6/5$ and $C_2 = 249/175$, which yields $P \sim 1 + 6\,\eta/5 + 249\,\eta^2/175$ and $\sqrt{P} \sim 1 + 3\,\eta/5 + 93\,\eta^2/175$.

Appendix 6

Time-harmonic flows defined by potentials of the form (3) and (6) have been considered in this study. Time-harmonic flows defined by the potentials

$$\mathfrak{Re}\ \phi(\mathbf{x})\,e^{-i f t} \quad \mathfrak{Re}\ \phi(\mathbf{x})\,e^{\varepsilon t - i f t} = \mathfrak{Re}\ \phi(\mathbf{x})\,e^{-i(f+i\varepsilon)t} \tag{66}$$

are also used in the literature. The complex frequency $f - i\,\varepsilon$ in (6) becomes $f + i\,\varepsilon$ in (66). The free-surface boundary condition (7) is modified as

$$\phi_z - f^2\phi + i\,2\tau\,\phi_x + F^2\phi_{xx} - 2\,i\,\varepsilon\,(f\phi - i\,F\phi_x) = 0\,.$$

The dispersion function (10) is unchanged if elementary waves

$$W(\mathbf{x}) = e^{-i(\alpha x + \beta y)} \cosh[\,k\,(z+d)\,]/\cosh(k\,d) \tag{67}$$

are considered instead of (9a). The elementary wave (15a) then becomes

$$e^{-i(\alpha x+\beta y)+\varepsilon(\alpha_1 x+\beta_1 y)} \cosh[\,(k + i\,\varepsilon\,k_1)(z+d)\,]\,/\cosh[(k+i\,\varepsilon\,k_1)\,d\,]\,.$$

Expressions (16) and (18) are modified as $\Delta = 0$, $k_1\Delta_k + \alpha_1\Delta_\alpha + \Delta_f = 0$, $\alpha_1\Delta_\alpha + \beta_1\Delta_\beta + \Delta_f = 0$ and the complex frequency $f - i\,\varepsilon$ in (20) becomes $f + i\,\varepsilon$. The proportionality factor P in (15c), expression (22a) for μ, and the elementary wave (22b) are modified as $P = -\Delta_f/(k\,\Delta_k)$, $\mu = -\text{sign}\Delta_f\ \text{sign}\Delta_k$,

$$e^{-i(\alpha x+\beta y)+\mu\,\varepsilon(\alpha x+\beta y)/Q} \cosh[\,k\,(1+i\,\mu\,\varepsilon/Q)(z+d\,)\,]/\cosh[\,k\,(1+i\,\mu\,\varepsilon/Q)\,d\,]\,. \tag{68}$$

This elementary wave is bounded in the farfield if $\mu\,\text{sign}(\alpha\,x + \beta\,y) < 0$. It follows that (24) and (25) are unchanged. Thus, the farfield radiation function $\mathcal{R}^F$ in (63a) is unchanged if time-harmonic flows and elementary waves defined by (66) and (67) are considered instead of (63b) and (62b). The elementary waves (56) and (57) become

$$\frac{2\,e^{-i(\alpha x+\beta y)+kz}}{1+\exp[\,\mu\,(\alpha\,x+\beta\,y+i\,k\,z)/\sigma]} = e^{-i(\alpha x+\beta y)+kz}\left(1-\mu\,\frac{\sinh(\varphi/\sigma)+i\sin(V/\sigma)}{\cosh(\varphi/\sigma)+\cos(V/\sigma)}\right). \tag{69}$$

Expression (59a) for the nearfield radiation function $\mathcal{R}^N$ is unchanged, but expression (59b) for the function Θ becomes

$$\Theta(\mathbf{x}) = \frac{\tanh(\varphi/\sigma) + i\,\tanh V\,\sin(V/\sigma)\,/\cosh(\varphi/\sigma)}{1+\cos(V/\sigma)\,/\cosh(\varphi/\sigma)} \quad \text{with} \quad \begin{Bmatrix} \varphi = \alpha\,x + \beta\,y \\ V = k\,(z+d\,) \end{Bmatrix}. \tag{70}$$

In the deep-water limit, Eq. 70 agrees with the expression given in [13]. The nearfield radiation function $\mathcal{R}^N$ defined by (59a) and (70) is identical to the function $\mathcal{R}^N$ obtained by formally performing the substitution $(f,\alpha,\beta) \to -(f,\alpha,\beta)$ in (59a) and (59b), as might be expected. Expression (63a) for the farfield radiation function $\mathcal{R}^F$ is unchanged by this substitution.

If time-harmonic flows defined by potentials of the form (3) and (6) are considered with elementary waves (67), the dispersion function (10) is modified as

$$\Delta = k\tanh(k\,d) - (f + F\alpha)^2. \tag{71}$$

Thus, (17) and (21) become $\Delta_f = -2\,(f + F\alpha)\,, \Delta_k = \tanh(k\,d) + k\,d/\cosh^2(k\,d) + 2\,(\tau + F^2\alpha)\,\alpha/k$ and the dispersion curves defined by the dispersion relation $\Delta = 0$ are the mirror image, with respect to the axis $\alpha = 0$, of the dispersion curves associated with (10). Expressions (16), (18) and (20) are unchanged. Expression (22a) is unchanged, but the elementary wave (22b) becomes (68). Thus, (25b) is modified as

$$\mathcal{R}^F = 1 - \operatorname{sign}\Delta_f \operatorname{sign}\Delta_k \operatorname{sign}(\alpha\,x + \beta\,y)\,. \tag{72}$$

The elementary waves (56) and (57) become (69), with μ given by (22a). Expression (59a) for the nearfield radiation function $\mathcal{R}^N$ is modified as

$$\mathcal{R}^N = 1 - \operatorname{sign}\Delta_f \operatorname{sign}\Delta_k\,\Theta, \tag{73}$$

where the function Θ is given by (70) instead of (59b). The farfield radiation function (72) and the nearfield radiation function given by (73) and (70) are identical to the functions $\mathcal{R}^F$ and $\mathcal{R}^N$ obtained by formally performing the substitution $(\alpha\,,\beta) \to -(\alpha\,,\beta)$ in (63a) and in (59a) and (59b), as might again be expected.

References

1. Witham GB (1974) Linear and nonlinear waves. Wiley-Interscience
2. Lighthill J (1978) Waves in Fluids. Cambridge University Press
3. John (1950) On the motion of floating bodies. Comm Pure Appl Math 3:45–101
4. Haskind MD (1956) The hydrodynamic theory of ship oscillations in rolling and pitching. Prikl Matematica i Mek 10:33–66
5. Peters AS, Stoker JJ (1957) The motion of a ship. Comm Pure Appl Math 10:399–490
6. Wehausen JV, Laitone EV (1960) Surface waves. Handbuch der Physics vol 9 Springer Verlag Berlin, pp 446–778
7. Kuznetsov N, Maz'ya V, Vainberg B (2002) Linear water waves: a mathematical approach. Cambridge Univ Press
8. Yang C, Löhner R, Noblesse F (2004) Comparison of classical and simple free-surface Green functions. IL J Offshore Polar Eng 14:257–264
9. Yang C, Löhner R, Noblesse F (2005) Comparison of classical and weakly-singular boundary-integral representations of free-surface potential flows IL J Offshore Polar Eng 15:241–248
10. Lighthill J (1967) On waves generated in dispersive systems by travelling forcing effects, with applications to the dynamics of rotating fluids. J Fluid Mech 27:725–752
11. Noblesse F (2001) Analytical representation of ship waves. Ship Techn Res 48:23–48
12. Lamb H (1932) Hydrodynamics Dover publications
13. Noblesse F, Yang C (2004) A simple Green function for diffraction-radiation of time-harmonic waves with forward speed. Ship Techn Res 51:35–52
14. Chen XB (2006) Analytical features of unsteady ship waves. In: Chwang AT, Teng MH, Valentine DT (eds) Advances in engineering mechanics, reflections and outlooks – In Honor of Theodore Yao-Tsu Wu. World Scientific
15. Chen XB, Noblesse F (1997) Dispersion relation and farfield waves. In: 12th Il Workshop Water Waves and Floating Bodies. Carry-le-Rouet, France, pp 31–34
16. Noblesse F, Chen XB (1995) Decomposition of free-surface effects into wave and near-field components. Ship Techn Res 42:167–185
17. Chen XB, Zhao R (2001) Steady free-surface flow in water of finite depth. In: 16th Il Workshop Water Waves and Floating Bodies. Hiroshima, Japan
18. Havelock TH (1908) The propagation of groups of waves in dispersive media, with applications to waves on water produced by a travelling disturbance. Proc R Soc London 89:489–499
19. Kinoshita M (1943) Wave resistance of a sphere in a shallow sea. J Soc Naval Architects Japan 73:19–37
20. Noblesse F, Yang C, Espinosa R, Löhner R (2006) Fundamental wavenumbers of diffraction-radiation by a ship that advances through regular waves in finite water depth. IL J Soc Offshore Polar Engng (ISOPE) 16:241–252